EXIT PATHS AND CONSTRUCTIBLE STACKS.

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Abstract. For a Whitney stratification S of a space X (or more generally a topological stratification in the sense of Goresky and MacPherson) we introduce the notion of an S-constructible stack of categories on X. The motivating example is the stack of S-constructible perverse sheaves. We introduce a 2-category $EP_{\leq 2}(X,S)$, called the exit-path 2-category, which is a natural stratified version of the fundamental 2-groupoid. Our main result is that the 2-category of S-constructible stacks on X is equivalent to the 2-category of 2-functors $2\mathrm{Funct}(EP_{\leq 2}(X,S),\mathbf{Cat})$ from the exit-path 2-category to the 2-category of small categories.

1. Introduction

This paper is concerned with a generalization of the following well-known and very old theorem:

Theorem 1.1. *Let X be a connected, locally contractible topological space. The category of locally constant sheaves of sets on X is equivalent to the category of G-sets, where G is the fundamental group of X.*

We wish to generalize this theorem in two directions. In one direction we will consider sheaves which are not necessarily locally constant – namely, constructible sheaves. In the second direction we will consider sheaves of "higher-categorical" objects – these generalizations of sheaves are usually called *stacks*. Putting these together, we get the "constructible stacks" of the title. In this paper, we introduce an object – the *exit-path 2-category* – which plays for constructible stacks the same role the fundamental group plays for locally constant sheaves.

1.1. **Exit paths and constructible sheaves.** A sheaf F on a space X is called "constructible" if the space may be decomposed into suitable pieces with F locally constant on each piece. To get a good theory one needs to impose some conditions on the decomposition – for our purposes the notion of a *topological stratification*, introduced in [7], is the most convenient. A topological stratification S of X is a decomposition of X into topological manifolds, called "strata," which are required to fit together in a nice way. (Topological stratifications are more general than Whitney stratifications and Thom-Mather stratifications [21], [16], and they are less general than Siebenmann's CS sets [20] and Quinn's manifold stratified spaces [19]. A precise definition is given in [7] and in section 3.) A sheaf is called *S-constructible* if it is locally constant along each stratum of (X,S).

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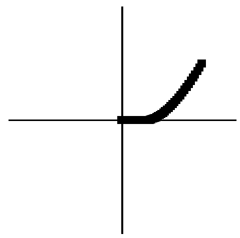
MacPherson observed (unpublished) that, for a fixed stratification S of X, it is possible to give a description of the S-constructible sheaves on X in terms of monodromy along certain paths. A path $\gamma : [0,1] \to X$ has the *exit property* with respect to S if, for each $t_1 \leq t_2 \in [0,1]$, the dimension of the stratum containing $\gamma(t_1)$ is less than or equal to the dimension of the stratum containing t_2. Here is a picture of an exit path in the plane, where the plane is stratified by the origin, the rays of the axes, and the interiors of the quadrants:



The concatenation of two exit paths is an exit path, and passing to homotopy classes yields a category we call $EP_{\leq 1}(X,S)$. That is, the objects of $EP_{\leq 1}(X,S)$ are points of X, and the morphisms are homotopy classes of exit paths between points. (We require that the homotopies $h : [0,1] \times [0,1] \to X$ have the property that each $h(t,-)$ is an exit path, and that h does not intersect strata in a pathological way. We conjecture that the latter "tameness" condition can be removed.) Then we have the following analog of theorem 1.1:

Theorem 1.2 (MacPherson)**.** *Let (X,S) be a topologically stratified space. The category of S-constructible sheaves of sets is equivalent to the category* $\mathrm{Funct}\big(EP_{\leq 1}(X,S), \mathbf{Sets}\big)$ *of* **Sets***-valued functors on $EP_{\leq 1}(X,S)$.*

Example 1.3. Let D be the open unit disk in the complex plane, and let S be the stratification by the origin $\{0\}$ and its complement $D - \{0\}$. Then $EP_{\leq 1}(D,S)$ is equivalent to a category with two objects, one labeled by 0 and one labeled by some other point $x \in D - \{0\}$. The arrows in this category are generated by arrows $\alpha : 0 \to x$ and $\beta : x \to x$; β generates the automorphism group $\mathbb{Z}$ and we have $\beta \circ \alpha = \alpha$. The map α is represented by an exit path from 0 to x in D and the path β is represented by a loop around 0 based at x.

It follows that an S-constructible sheaf (of, say, complex vector spaces) on D is given by two vector spaces V and W, a morphism $a : V \to W$ and a morphism $b : W \to W$, with the property that b is invertible and that $b \circ a = a$.

Example 1.4. Let $\mathbb{P}^1 = \mathbb{C} \cup \infty$ be the Riemann sphere, and let S be the stratification of $\mathbb{P}^1$ by one point $\{\infty\}$ and the complement $\mathbb{C}$. Then $EP_{\leq 1}(\mathbb{P}^1, S)$ is equivalent to a category with two objects, one labeled by ∞ and the other labeled by $0 \in \mathbb{C}$. The only nontrivial arrow in this category is represented by an exit path from ∞ to 0; all such paths are homotopic to each other.

It follows that an S-constructible sheaf on $\mathbb{P}^1$ is given by two vector spaces V and W, and a single morphism $V \to W$.

1.2. **Perverse sheaves.** Let (X,S) be a topologically stratified space. For each function $p : S \to \mathbb{Z}$ from connected strata of (X,S) to integers, there is an abelian category $\mathbf{P}(X,S,p)$ of "S-constructible perverse sheaves on X of perversity p," introduced in [1]. It is a full subcategory of the derived category of sheaves on X; its objects are complexes of sheaves whose

cohomology sheaves are S-constructible, and whose derived restriction and corestriction to strata satisfy certain cohomology vanishing conditions depending on p. It is difficult to lay hands on the objects and especially the morphisms of $\mathbf{P}(X, S, p)$, although we get $\mathrm{Sh}_S(X)$ as a special case when p is constant.

There is a small industry devoted to finding concrete descriptions of the category $\mathbf{P}(X, S, p)$ in terms of "linear algebra data," similar to the description of S-constructible sheaves given in the examples above ([15], [5], [3], [2], [25]). Here (X, S) is usually a complex analytic space, with complex analytic strata, and p is the "middle perversity" which associates to each stratum its complex dimension. The first example was found by Deligne:

Example 1.5. Let D and S be as in example 1.3. The category $\mathbf{P}(X, S, p)$, where p is the middle perversity, is equivalent to the category of tuples (V, W, m, n), where V and W are $\mathbb{C}$-vector spaces, $m : V \to W$ and $n : W \to V$ are linear maps, and $1_W - mn$ is invertible.

A topological interpretation of this description was given by MacPherson and Vilonen [15]. If (L, S) is a compact topologically stratified space then the open cone CL on L has a natural topological stratification T, in which the cone point is a new stratum. MacPherson and Vilonen gave a description of perverse sheaves on (CL, T) in terms of perverse sheaves on L, generalizing Deligne's description 1.5.

One of the important properties of perverse sheaves is that they form a *stack*; it means that a perverse sheaf on a space X may be described in the charts of an open cover of X. A topologically stratified space has an open cover in which the charts are of the form $CL \times \mathbb{R}^k$. The stack property together with the MacPherson-Vilonen construction give an inductive strategy for computing categories of perverse sheaves. One of the motivations for the theory in this paper is to analyze this strategy systematically; see [23].

1.3. **Constructible stacks.** In this paper, we introduce the notion of a *constructible stack* on a topologically stratified space. Our main example is the stack $\mathcal{P}$ of S-constructible perverse sheaves discussed in section 1.2. Our main result is a kind of classification of constructible stacks, analogous to the description of constructible sheaves by exit paths.

Main Theorem (Theorem 7.14)**.** Let (X, S) be a topologically stratified space. There is a 2-category $EP_{\leq 2}(X, S)$, introduced in section 7, such that the 2-category of S-constructible stacks on X is equivalent to the 2-category of 2-functors $2\mathrm{Funct}(EP_{\leq 2}(X, S), \mathbf{Cat})$.

The appearance of 2-categories in this theorem is an application of a well-known philosophy of Grothendieck [9]. It is a modification of theorems in [18] and [22], where it was shown that *locally constant* stacks on X correspond to representations of higher *groupoids*, namely the groupoid of points, paths, homotopies, homotopies between homotopies, and so on in X. $EP_{\leq 2}(X)$ is a 2-truncated, stratified version of this: the objects are the points of X, the morphisms are exit paths, and the 2-morphisms are homotopy classes of homotopies between exit paths. (Once again, we require a tameness condition on our homotopies and also our homotopies between homotopies.)

Example 1.6. Let $\mathbb{P}^1$ and S be as in example 1.4. Then $EP_{\leq 2}(\mathbb{P}^1, S)$ is equivalent to a 2-category with two objects, labeled by ∞ and 0 as before, and one arrow from ∞ to 0

represented by an exit path α. The group of homotopies from α to itself is $\mathbb{Z}$, generated by a homotopy that rotates α around the 2-sphere once.

It follows that an S-constructible stack on $\mathbb{P}^1$ is given by a pair $\mathbf{C}_\infty$ and $\mathbf{C}_0$ of categories, a functor $\alpha : \mathbf{C}_\infty \to \mathbf{C}_0$, and a natural automorphism $f : \alpha \to \alpha$. For the stack of S-constructible perverse sheaves, $\mathbf{C}_0$ is the category of vector spaces, $\mathbf{C}_\infty$ is Deligne's category described in example 1.5, α is the forgetful functor $\alpha : (V, W, m, n) \mapsto W$, and f is the map $1_W - mn$, which is invertible by assumption.

1.4. **Notation and conventions.** $\mathbb{R}$ denotes the real numbers. For $a, b \in \mathbb{R}$ with $a \leq b$ we use (a, b) to denote the open interval and $[a, b]$ to denote the closed interval between a and b. We use $[a, b)$ and $(a, b]$ to denote half-open intervals. If L is a compact space then CL denotes the open cone on L, that is, the space $L \times [0, 1)/L \times \{0\}$. A d-cover of a space X is a collection of open subsets of X that covers X and that is closed under finite intersections.

For us, a "2-category" is a strict 2-category in the sense that composition of 1-morphisms is strictly associative. On the other hand we use "2-functor" to refer to morphisms of 2-categories that only preserve composition of 1-morphisms up to isomorphism. We will refer to sub-2-categories of a 2-category $\mathbf{C}$ as simply "subcategories of $\mathbf{C}$." For more basic definitions and properties of 2-categories see appendix B.

All our stacks are stacks of categories. We write $\mathrm{Prest}(X)$ for the 2-category of prestacks on a space X and $\mathrm{St}(X) \subset \mathrm{Prest}(X)$ for the full subcategory of $\mathrm{Prest}(X)$ whose objects are stacks. We use $\mathrm{St}_{lc}(X) \subset \mathrm{St}(X)$ to denote the full subcategory of locally constant stacks on X, which we introduce in section 2. When S is a topological stratification (definition 3.1) of X we write $\mathrm{St}_S(X) \subset \mathrm{St}(X)$ for the full subcategory of S-constructible stacks on X, which we introduce in section 3. For more basic definitions and properties of stacks see appendix A.

2. Locally constant stacks

In this section we introduce locally constant stacks of categories. A stack is called *constant* if it equivalent to the stackification of a constant prestack, and *locally constant* if this is true in the charts of an open cover. Our main objective is to give an equivalent definition that is easier to check in practice: on a locally contractible space, a stack $\mathcal{C}$ is locally constant if and only if the restriction functor $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence of categories whenever V and U are contractible. This is theorem 2.9. We also develop some basic properties of locally constant stacks, including a base-change result (theorem 2.11) and the homotopy invariance of the 2-category of locally constant stacks (theorem 2.7).

2.1. **Constant stacks.** Let $\mathbf{C}$ be a small category. On any space X we have the constant $\mathbf{C}$-valued prestack, and its stackification. We will denote the prestack by $\mathbf{C}_{p;X}$, and its stackification by $\mathbf{C}_X$.

Example 2.1. Let X be a locally contractible space, or more generally any space in which each point has a fundamental system of neighborhoods over which each locally constant sheaf is constant. If $\mathbf{C}$ is the category of sets, then $\mathbf{C}_X$ is naturally equivalent to the stack $\mathcal{LC}_X$ of locally constant sheaves. That is, the map $\mathbf{C}_{p;X} \to \mathcal{LC}_X$ that takes a set $E \in \mathbf{C} = \mathbf{C}_{p;X}(U)$

to the constant sheaf over U with fiber E induces an equivalence $\mathbf{C}_X \to \mathcal{LC}_X$. Indeed, it induces an equivalence on stalks by the local contractibility of X.

Proposition 2.2. *Let X be a locally contractible space and $\mathbf{C}$ be a small category.*

1. *If F and G are objects in $\mathbf{C}_X(X)$, then the sheaf $\underline{\mathrm{Hom}}(F,G) : U \mapsto \mathrm{Hom}(F|_U, G|_U)$ is locally constant on X.*
2. *Let $U \subset X$ be an open set. For each point $x \in U$, the restriction functor from $\mathbf{C} = \mathbf{C}_{p;X}(U)$ to the stalk of $\mathbf{C}_X$ at x is an equivalence of categories.*
3. *Let $U \subset X$ be an open set. Suppose that U is contractible. Then for each point $x \in U$, the restriction functor $\mathbf{C}_X(U) \to \mathbf{C}_{X,x}$ is an equivalence of categories.*

Proof. The map of prestacks $\mathrm{Hom} : \mathbf{C}_{p;X}^{op} \times \mathbf{C}_{p;X} \to \mathbf{Sets}_{p;X}$ induces a map of stacks $\mathbf{C}_X^{op} \times \mathbf{C}_X \to \mathbf{Sets}_X \cong \mathcal{LC}_X$. The object in $\mathcal{LC}_X(X)$ associated to a pair $(F,G) \in \big(\mathbf{C}_X^{op} \times \mathbf{C}_X\big)(X)$ is exactly the sheaf $\underline{\mathrm{Hom}}(F,G)$. This proves the first assertion.

The second assertion is trivial. To prove the third assertion, note that $\mathbf{C}_X(U) \to \mathbf{C}_{X,x}$ is always essentially surjective, since the equivalence $\mathbf{C}_{p;X}(U) \to \mathbf{C}_{X,x}$ factors through it. We therefore only have to show that $\mathbf{C}_X(U) \to \mathbf{C}_{X,x}$ is fully faithful. For objects F and G of $\mathbf{C}_X(U)$, we have just seen that $\underline{\mathrm{Hom}}(F,G)$ is locally constant on U. Since U is contractible $\underline{\mathrm{Hom}}(F,G)$ is constant, and so $\mathrm{Hom}_{\mathbf{C}_X(U)}(F,G) = \underline{\mathrm{Hom}}(F,G)(U) \to \underline{\mathrm{Hom}}(F,G)_x \cong \mathrm{Hom}(F_x, G_x)$ is a bijection. This completes the proof. □

2.2. Locally constant stacks.

Definition 2.3. A stack $\mathcal{C}$ on X is called *locally constant* if there exists an open cover $\{U_i\}_{i\in I}$ of X such that $\mathcal{C}|_{U_i}$ is equivalent to a constant stack. Let $\mathrm{St}_{lc}(X) \subset \mathrm{St}(X)$ denote the full subcategory of the 2-category of stacks on X whose objects are the locally constant stacks.

Proposition 2.4. *Let X be a locally contractible space and let $\mathcal{C}$ be a locally constant stack on X.*

1. *Let $U \subset X$ be an open set, and let $F, G \in \mathcal{C}(U)$. The sheaf $\underline{\mathrm{Hom}}(F,G)$ is locally constant on U.*
2. *Every point $x \in X$ has a contractible neighborhood V such that the restriction map $\mathcal{C}(V) \to \mathcal{C}_x$ is an equivalence of categories.*

Proof. Assertion 1 follows directly from assertion 1 of proposition 2.2, and assertion 2 follows directly from assertion 3 of proposition 2.2. □

Proposition 2.5. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Suppose $\mathcal{C}$ is locally constant on Y. Then $f^*\mathcal{C}$ is locally constant on X.*

Proof. If $\mathcal{C}$ is constant over the open sets U_i of Y, then $f^*\mathcal{C}$ will be constant over the open sets $f^{-1}(U_i)$ of X. □

The homotopy invariance of $\mathrm{St}_{lc}(X)$ is a consequence of the following base-change result:

Proposition 2.6. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Let g denote the map $(id, f) : [0,1] \times X \to [0,1] \times Y$, and let $p : [0,1] \times X \to X$ and*

$q : [0,1] \times Y \to Y$ be the natural projection maps.

$$\begin{tikzcd} {[0,1] \times X} \arrow[r, "g"] \arrow[d, "p"'] & {[0,1] \times Y} \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Let $\mathcal{C}$ be a locally constant stack on $[0,1] \times Y$. Then the base change map $f^* q_* \mathcal{C} \to p_* g^* \mathcal{C}$ is an equivalence of stacks on X.

Proof. Let us first prove the following claim: every point $t \in [0,1]$ has a neighborhood $I \subset [0,1]$ such that $\mathcal{C}\big([0,1] \times Y\big) \to \mathcal{C}\big(I \times Y\big)$ is an equivalence of categories. There is an open cover of $[0,1] \times Y$ of the form $\{I_\alpha \times U_\beta\}$ such that $\mathcal{C}$ is constant over each chart $I_\alpha \times U_\beta$. By by basic properties of the interval, $\mathcal{C}$ is constant over $[0,1] \times U_\beta$, and for any subinterval $I \subset [0,1]$ the restriction map $\mathcal{C}\big([0,1] \times U_\beta\big) \to \mathcal{C}(I \times U_\beta)$ is an equivalence of categories. This implies the claim.

Let y be a point in Y, and let t be a point in $[0,1]$. Let $\{U\}$ be a fundamental system of neighborhoods of y. According to the claim, we may pick for each U an open set $I_U \subset [0,1]$ such that the restriction functor $\mathcal{C}\big([0,1] \times U\big) \to \mathcal{C}(I_U \times U)$ is an equivalence of categories. We may choose the I_U in such a way that the open sets $I_U \times U \subset [0,1] \times Y$ form a fundamental system of neighborhoods of (t,y). It follows that the natural restriction functor on stalks $(q_* \mathcal{C})_y \to \mathcal{C}_{(t,y)}$ is an equivalence.

Now let x be a point in X. We have natural equivalences

$$\begin{array}{ccccc} (f^* q_* \mathcal{C})_x & \cong & (q_* \mathcal{C})_{f(x)} & \cong & \mathcal{C}_{(t,f(x))} \\ (p_* g^* \mathcal{C})_x & \cong & (g^* \mathcal{C})_{(t,x)} & \cong & \mathcal{C}_{(t,f(x))} \end{array}$$

This completes the proof. $\square$

Theorem 2.7 (Homotopy invariance)**.** *Let X be a topological space, and let π denote the projection map $[0,1] \times X \to X$. Then π_* and π^* are inverse equivalences between the 2-category of locally constant stacks on X, and the 2-category of locally constant stacks on $[0,1] \times X$.*

Proof. Let $\mathcal{C}$ be a locally constant stack on X and let $\mathcal{D}$ be a locally constant stack on $I \times X$.

Let $x \in X$, let i_x denote the inclusion map $\{x\} \hookrightarrow X$, let j_x denote the inclusion map $[0,1] \cong [0,1] \times x \hookrightarrow [0,1] \times X$, and let p denote the map $[0,1] \to \{x\}$. By proposition 2.6, the natural map $(\pi_* \pi^* \mathcal{C})_x = i_x^* \pi_* \pi^* \mathcal{C} \to p_* j_x^* \pi^* \mathcal{C}$ is an equivalence. But $p_* j_x^* \pi^* \mathcal{C} \cong p_* p^* i_x^* \mathcal{C} = p_* p^* (\mathcal{C}_x)$. The natural map $\mathcal{C}_x \to (\pi_* \pi^* \mathcal{C})_x \cong p_* p^* (\mathcal{C}_x)$ coincides with the adjunction map $\mathcal{C}_x \to p_* p^* (\mathcal{C}_x)$. Since $p^* \mathcal{C}_x$ is constant, $\mathcal{C}_x \to p_* p^* (\mathcal{C}_x)$ is an equivalence by proposition 2.2. It follows that $\mathcal{C} \to \pi_* \pi^* \mathcal{C}$ is an equivalence of stacks.

Now let $(t,x) \in [0,1] \times X$. There is an equivalence $(\pi^* \pi_* \mathcal{D})_{(t,x)} \cong (\pi_* \mathcal{D})_x$. Once again proposition 2.6 provides an equivalence $(\pi_* \mathcal{D})_x \cong p_* j_x^* \mathcal{D} = j_x^* \mathcal{D}\big([0,1]\big)$. The locally constant stack $j_x^* \mathcal{D}$ is constant on $[0,1]$, so $j_x^* \mathcal{D}\big([0,1]\big) \cong \mathcal{D}_{(t,x)}$ by proposition 2.2. It follows that $\pi^* \pi_* \mathcal{D} \to \mathcal{D}$ is an equivalence of stacks, completing the proof.

$\square$

Corollary 2.8. *Let X and Y be topological spaces, and let $f : X \to Y$ be a homotopy equivalence.*

1. *The 2-functor $f^* : \mathrm{St}_{lc}(Y) \to \mathrm{St}_{lc}(X)$ is an equivalence of 2-categories.*
2. *Let $\mathcal{C}$ be a locally constant stack on Y. Then the natural functor $\mathcal{C}(Y) \to f^*\mathcal{C}(X)$ is an equivalence of categories.*

Proof. Let $g : Y \to X$ be a homotopy inverse to f, and let $H : [0,1] \times X \to X$ be a homotopy between $g \circ f$ and 1_X. Let π denote the projection map $[0,1] \times X \to X$, and for $t \in [0,1]$ let i_t denote the map $X \to [0,1] \times X : x \mapsto (t, x)$. Then $i_t^* \cong \pi_*$ by theorem 2.7. It follows that $i_0^* \cong i_1^*$, and that $(H \circ i_0)^* \cong (H \circ i_1)^*$. But $H \circ i_0 = g \circ f$ and $H \circ i_1 = 1_X$, so $f^* \circ g^* \cong 1_{\mathrm{St}_{lc}(X)}$. Similarly using a homotopy $G : I \times Y \to Y$ we may construct an equivalence $g^* \circ f^* \cong 1_{\mathrm{St}_{lc}(Y)}$. This proves the first assertion.

To prove the second assertion, let $*$ denote the trivial category. By (1), $\mathrm{Hom}_{\mathrm{St}(Y)}(*, \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{St}(X)}\big(*, f^*\mathcal{C}\big)$. But $\mathcal{C}(Y) \cong \mathrm{Hom}(*, \mathcal{C})$ and $f^*\mathcal{C}(X) \cong \mathrm{Hom}(*, f^*\mathcal{C})$. This completes the proof. □

Theorem 2.9. *Let X be a locally contractible space, and let $\mathcal{C}$ be a stack of categories on X. The following are equivalent:*

1. *$\mathcal{C}$ is locally constant.*
2. *If U and V are two open subsets of X with $V \subset U$, and the inclusion map $V \hookrightarrow U$ is a homotopy equivalence, then the restriction functor $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence of categories.*
3. *If U and V are any two contractible open subsets of X, and $V \subset U$, then $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence of categories.*
4. *There exists a collection $\{U_i\}$ of contractible open subsets of X such that each point $x \in X$ has a fundamental system of neighborhoods of the form U_i, and such that $\mathcal{C}(U_i) \to \mathcal{C}(U_j)$ is an equivalence of categories whenever $U_j \subset U_i$.*

Proof. Suppose $\mathcal{C}$ is locally constant, and let U and V be as in condition (2). Then corollary 2.8 implies that the restriction functor $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence of categories, so condition (1) implies condition (2). Clearly condition (2) implies condition (3), and condition (3) implies condition (4). Let us show condition (4) implies condition (1).

Suppose $\mathcal{C}$ satisfies condition (4). To show $\mathcal{C}$ is locally constant it is enough to show its restriction to each of the distinguished charts in $\{U_i\}$ is constant. Let $U \subset X$ be such a chart. Since each point $x \in U$ has a fundamental system of neighborhoods $\{V\}$ from $\{U_i\}$, and since $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence for each V, the map $\mathcal{C}(U) \to \mathcal{C}_x$ is an equivalence for each U. It follows that the natural map from the constant stack $\big(\mathcal{C}(U)\big)_U$ to $\mathcal{C}|_U$ is an equivalence on stalks, and therefore an equivalence. □

2.3. **Direct images and base change.** Let f be a continuous map between locally contractible spaces. As an application of theorem 2.9, we may give easy proofs of some basic properties of the direct image f_* of locally constant stacks.

Proposition 2.10. *Let X and Y be locally contractible spaces. Let $f : X \to Y$ be a locally trivial fiber bundle, or more generally a Serre fibration. Let $\mathcal{C}$ be a locally constant stack on X. Then $f_*\mathcal{C}$ is locally constant on Y.*

Proof. By theorem 2.9, it suffices to show that the restriction functor $f_*\mathcal{C}(U) \to f_*\mathcal{C}(V)$, which is equal to the restriction functor $\mathcal{C}\big(f^{-1}(U)\big) \to \mathcal{C}\big(f^{-1}(V)\big)$, is an equivalence of categories whenever $V \subset U \subset X$ are open sets and U and V are contractible. Since f is a Serre fibration, the inclusion map $f^{-1}(V) \hookrightarrow f^{-1}(U)$ is a homotopy equivalence, and the proposition follows from corollary 2.8. $\square$

Theorem 2.11. *Let X and S be locally contractible spaces. Let $p : X \to S$ be a locally trivial fiber bundle, or more generally a Serre fibration. Let T be another locally contractible space, and let $f : T \to S$ be any continuous function. Set $Y = X \times_S T$, and let $g : Y \to X$ and $q : Y \to T$ denote the projection maps.*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

Let $\mathcal{C}$ be a locally constant stack on X. The base-change map $f^*p_*\mathcal{C} \to q_*g^*\mathcal{C}$ is an equivalence of stacks.

Proof. The statement is local on T and S, so we may assume both T and S are contractible. Since p and q are Serre fibrations the stacks $f^*p_*\mathcal{C}$ and $q_*g^*\mathcal{C}$ are locally constant, and therefore constant. To show that the base-change map is an equivalence of stacks it is enough to show that the functor $f^*p_*\mathcal{C}(T) \to q_*g^*\mathcal{C}(T)$ is an equivalence of categories. We have $q_*g^*\mathcal{C}(T) = g^*\mathcal{C}(Y)$, which by corollary 2.8 is equivalent to $\mathcal{C}(X)$. Furthermore, corollary 2.8 shows that $f^*p_*\mathcal{C}(T) \cong p_*\mathcal{C}(S) = \mathcal{C}(X)$. This completes the proof. $\square$

3. Constructible stacks

In this section we introduce constructible stacks. First we review the topologically stratified spaces of [7]. The definition is inductive: roughly, a stratification S of a space X is a decomposition into pieces called "strata," such that the decomposition looks locally like the cone on a simpler (lower-dimensional) stratified space. A stack on X is called "S-constructible" if its restriction to each stratum is locally constant. This definition is somewhat unwieldy, and we give a more usable criterion in theorem 3.13, analogous to theorem 2.9 for locally constant stacks: a stack is S-constructible if and only if the restriction from a "conical" open set to a smaller conical open set is an equivalence of categories. This criterion is a consequence of a stratified-homotopy invariance statement (corollary 3.12).

3.1. Topologically stratified spaces.

Definition 3.1. Let X be a paracompact Hausdorff space.

A 0-dimensional topological stratification of X is a homeomorphism between X and a countable discrete set of points. For $n > 0$, an n-dimensional *topological stratification* of X is a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

of X by closed subsets X_i, such that for each i and for each point $x \in X_i - X_{i-1}$, there exists a neighborhood U of x, a compact Hausdorff space L, an $(n-i-1)$-dimensional topological

stratification

$$\emptyset = L_{-1} \subset L_1 \subset L_2 \subset \ldots \subset L_{n-i-1} = L$$

of L, and a homeomorphism $CL \times \mathbb{R}^i \cong U$ that takes each $CL_j \times \mathbb{R}^i$ homeomorphically to $U \cap X_j$. Here $CL = [0,1) \times L/\{0\} \times L$ is the open cone on L if L is non-empty; if L is empty then let CL be a one-point space.

A finite dimensional *topologically stratified space* is a pair (X, S) where X is a paracompact Hausdorff space and S is an n-dimensional topological stratification of X for some n.

Let (X, S) be a topologically stratified space with filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

Note the following immediate consequences of the definition:

1. If $X_i - X_{i-1}$ is not empty, then it is an i-dimensional topological manifold.
2. If $U \subset X$ is open then the filtration $U_{-1} \subset U_0 \subset U_1 \subset \ldots$ of U, where $U_i = U \cap X_i$, is a topological stratification.

We will call the connected components of $X_i - X_{i-1}$, or unions of them, i-dimensional *strata*. We will call the neighborhoods U homeomorphic to cones "conical neighborhoods":

Definition 3.2. Let (X, S) be an n-dimensional topologically stratified space. An open set $U \subset X$ is called a *conical open subset* of X if U is homeomorphic to $CL \times \mathbb{R}^i$ for some L as in definition 3.1.

Remark 3.3. By definition, every point in a topologically stratified space has an conical neighborhood $CL \times \mathbb{R}^k$. One of the quirks of topological stratifications (as opposed to e.g. Whitney stratifications) is that the space L is not uniquely determined up to homeomorphism: there even exist non-homeomorphic manifolds L_1 and L_2 such that $CL_1 \cong CL_2$ (see [17]).

The following definition, from [7], is what is usually meant by "stratified map."

Definition 3.4. Let (X, S) and (Y, T) be topologically stratified spaces. A continuous map $f : X \to Y$ is *stratified* if it satisfies the following two conditions:

1. For any connected component C of any stratum $Y_k - Y_{k-1}$, the set $f^{-1}(C)$ is a union of connected components of strata of X.
2. For each point $y \in Y_i - Y_{i-1}$ there exists a neighborhood U of x in Y_i, a topologically stratified space
$$F = F_k \supset F_{k-1} \supset \ldots \supset F_{-1} = \emptyset$$
and a filtration-preserving homeomorphism
$$F \times U \cong f^{-1}(U)$$
that commutes with the projection to U.

We need a much broader definition:

Definition 3.5. Let (X, S) and (Y, T) be topologically stratified spaces. A continuous map $f : X \to Y$ is called *stratum-preserving* if for each k, and each connected component $Z \subset X_k - X_{k-1}$, the image $f(Z)$ is contained in $Y_\ell - Y_{\ell-1}$ for some ℓ.

Definition 3.6. Let (X, S) and (Y, T) be topologically stratified spaces, and let f and g be two stratum-preserving maps from X to Y. We say f and g are *homotopic relative to the stratifications* if there exists a homotopy $H : [0, 1] \times X \to Y$ between f and g such that the map $H(t, -) : X \to Y$ is stratum-preserving for every $t \in [0, 1]$.

A slightly irritating feature of this definition is that the space $[0, 1] \times X$ cannot be stratified without treating the boundary components $\{0\} \times X$ and $\{1\} \times X$ differently. We may take care of this by using the open interval: if (X, S) and (Y, T) are topologically stratified spaces, then we may endow $(0, 1) \times X$ with a topological stratification by setting $\big((0, 1) \times X\big)_i = (0, 1) \times X_{i-1}$. Note the following

1. Let $H : [0, 1] \times X \to Y$ be a stratified homotopy. The restriction of this map to $(0, 1) \times X$ is stratum-preserving.
2. Let f and g be two stratum-preserving maps. Then f and g are homotopic relative to the stratifications if and only if there exists a stratum-preserving map $H : (0, 1)\times X \to Y$ such that $f(-) = H(t_0, -)$ and $g(-) = H(t_1, -)$ for some $t_0, t_1 \in (0, 1)$.

Definition 3.7. Let (X, S) and (Y, T) be topologically stratified spaces. Let $f : X \to Y$ be a stratum-preserving map. Call f a *stratified homotopy equivalence* if there is a stratum-preserving map $Y \to X$ such that the composition $g \circ f$ is stratified homotopic to the identity map 1_X of X, and $f \circ g$ is stratified homotopic to the identity map 1_Y of Y.

Note that a "stratified homotopy equivalence" f need not be a stratified map in the sense of definition 3.4, but only stratum-preserving.

3.2. **Constructible stacks.**

Definition 3.8. Let (X, S) be a topologically stratified space and let $\mathcal{C}$ be a stack on X. $\mathcal{C}$ is called *constructible* with respect to S if, for each k, $i_k^*\mathcal{C}$ is locally constant on $X_k - X_{k-1}$, where $i_k : X_k - X_{k-1} \hookrightarrow X$ denotes the inclusion of the k-dimensional stratum into X.

Let $\mathrm{St}_S(X)$ denote the full subcategory of the 2-category $\mathrm{St}(X)$ of stacks on X whose objects are the S-constructible stacks.

The pullback of a constructible stack is constructible:

Proposition 3.9. *Let (X, S) and (Y, T) be two topologically stratified spaces. Let $f : X \to Y$ be a stratum-preserving map. If $\mathcal{C}$ is a T-constructible stack on Y, then $f^*\mathcal{C}$ is S-constructible on X.*

Proof. We have to show that $f^*\mathcal{C}$ is locally constant on $X_k - X_{k-1}$. It is enough to show it is locally constant on each connected component. Let C be a component of $X_k - X_{k-1}$, and let $i : C \to X$ be the inclusion. Then $i^*f^*\mathcal{C} \cong (f \circ i)^*\mathcal{C}$. But $f \circ i : C \to Y$ factors through $j : Y_\ell - Y_{\ell-1} \to Y$ for some ℓ, so $i^*f^*\mathcal{C}$ is obtained from pulling back $j^*\mathcal{C}$ on $Y_\ell - Y_{\ell-1}$ to C. By proposition 2.5, this is locally constant on C. $\square$

Proposition 3.10. *Let (X, S) be a topologically stratified space, and let C be a connected stratum of X. Let $i : C \hookrightarrow X$ denote the inclusion map. Let $p : (0, 1) \times C \to C$ and $q : (0, 1) \times X \to X$ denote the projection maps, and let j denote the inclusion map (id, i) :*

$(0,1)\times C \hookrightarrow X$.

$$\begin{tikzcd}
(0,1)\times C \arrow[r, "j"] \arrow[d, "p"'] & (0,1)\times X \arrow[d, "q"] \\
C \arrow[r, "i"'] & X
\end{tikzcd}$$

Endow $(0,1)\times X$ *with a topological stratification by setting* $\big((0,1)\times X\big)_k = (0,1)\times X_{k-1}$. *Let* $\mathcal{C}$ *be a stack on* $(0,1)\times X$ *constructible with respect to this stratification. Then the base-change map* $i^*q_*\mathcal{C} \to p_*j^*\mathcal{C}$ *is an equivalence of stacks.*

Proof. Let x be a point in X and let t be a point in $(0,1)$. As in the proof of proposition 2.6, we may show that the natural map $(q_*\mathcal{C})_x \to \mathcal{C}_{(t,x)}$ is an equivalence of categories. If x lies in the stratum C, then we have equivalences of categories:

$$\begin{array}{ccccc}
(i^*q_*\mathcal{C})_x & \cong & (q_*\mathcal{C})_x & \cong & \mathcal{C}_{(t,x)} \\
(p_*j^*\mathcal{C})_x & \cong & (j^*\mathcal{C})_{(t,x)} & \cong & \mathcal{C}_{(t,x)}
\end{array}$$

The base-change map commutes with these, proving the proposition. □

Theorem 3.11 (Homotopy invariance)**.** *Let* (X,S) *be a topologically stratified space. Endow* $(0,1)\times X$ *with a topological stratification* T *by setting* $\big((0,1)\times X\big)_i = (0,1)\times X_{i-1}$. *Let* π *be the stratified projection map* $(0,1)\times X \to X$. *The adjoint 2-functors* π_* *and* π^* *induce an equivalence between the 2-category of* S*-constructible stacks on* X *and the 2-category of* T*-constructible stacks on* $(0,1)\times X$.

Proof. We have to show the maps $\mathcal{C}\to\pi_*\pi^*\mathcal{C}$ and $\pi^*\pi_*\mathcal{D}\to\mathcal{D}$ are equivalences, where $\mathcal{C}$ is a constructible stack on X and $\mathcal{D}$ is a constructible stack on $(0,1)\times X$.

For each k, let i_k denote the inclusion map $X_k - X_{k-1}\hookrightarrow X$. To prove that $\mathcal{C}\to\pi_*\pi^*\mathcal{C}$ is an equivalence it suffices to show that $i_k^*\mathcal{C}\to i_k^*\pi_*\pi^*\mathcal{C}$ is an equivalence for each k. Let j_k denote the inclusion $(0,1)\times(X_k - X_{k-1})\hookrightarrow(0,1)\times X$, and let p_k denote the projection map $(0,1)\times(X_k-X_{k-1})\to X_k - X_{k-1}$.

$$\begin{tikzcd}
(0,1)\times\big(X_k - X_{k-1}\big) \arrow[r, "j_k"] \arrow[d, "p_k"'] & (0,1)\times X \arrow[d, "\pi"] \\
X_k - X_{k-1} \arrow[r, "i_k"] & X
\end{tikzcd}$$

By proposition 3.10, we have an equivalence $i_k^*\pi_*\pi^*\mathcal{C}\cong p_{k*}p_k^*i_k^*\mathcal{C}$. Since $i_k^*\mathcal{C}$ is locally constant on $X_k - X_{k-1}$, the map $i_k^*\mathcal{C}\to p_{k*}p_k^*i^*\mathcal{C}$ is an equivalence by theorem 2.7.

To show $\pi^*\pi_*\mathcal{D}\to\mathcal{D}$ is an equivalence, it is enough to show that for each k the map $j_k^*\pi^*\pi_*\mathcal{D}\to\mathcal{D}$ is an equivalence. By proposition 3.10 we have $j_k^*\pi^*\pi_*\mathcal{D}\cong p_k^*i_k^*\pi_*\mathcal{D}\cong p_k^*p_{k*}j_k^*\mathcal{D}$, and since $j_k^*\mathcal{D}$ is locally constant the map $p_k^*p_{k*}j_k^*\mathcal{D}\to j_k^*\mathcal{D}$ is an equivalence of stacks by theorem 2.7. □

Corollary 3.12. *Let* (X,S) *and* (Y,T) *be topologically stratified spaces, and let* $f: X\to Y$ *be a stratified homotopy equivalence.*

(1) *The 2-functor* $f^*:\mathrm{St}_T(Y)\to\mathrm{St}_S(X)$ *is an equivalence of 2-categories.*

(2) *Let $\mathcal{C}$ be an S-constructible stack on X. The functor $\mathcal{C}(X) \to f^*\mathcal{C}(Y)$ is an equivalence of categories.*

Proof. A proof identical to the one of corollary 2.8 gives both statements. $\square$

Theorem 3.13. *Let (X, S) be a topologically stratified space and let $\mathcal{C}$ be a stack on X. The following are equivalent:*

(1) *$\mathcal{C}$ is constructible with respect to the stratification.*
(2) *If U and V are two open subsets of X with $V \subset U$, and if the inclusion map $V \hookrightarrow U$ is a stratified homotopy equivalence, then the restriction functor $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence of categories.*
(3) *Whenever U and V are conical open subsets of X such that $V \subset U$ and the inclusion map $V \to U$ is a stratified homotopy equivalence, the restriction functor $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence of categories.*

If $\mathcal{C}$ satisfies these conditions then the natural functor $\mathcal{C}(U) \to \mathcal{C}_x$ is an equivalence of categories whenever U is a conical open neighborhood of x.

Proof. Suppose $\mathcal{C}$ is constructible, and let U and V be as in (2). Then $\mathcal{C}(U) \to \mathcal{C}(V)$ is an equivalence by corollary 3.12, so (1) implies (2). Clearly (2) implies (3).

Suppose now that $\mathcal{C}$ satisfies the third condition. Let $Y = X_k - X_{k-1}$ be a stratum, and let $i : Y \hookrightarrow X$ denote the inclusion map. Let $\{U\}$ be a collection of conical open sets in X that cover Y, and such that each $U \cap Y$ is closed in U. To show that $i^*\mathcal{C}$ is locally constant on Y it is enough to show that $j^*(\mathcal{C}|_U)$ is constant on $Y \cap U$, where j denotes the inclusion $Y \cap U \to U$.

For each $\epsilon > 0$ let $C_\epsilon L \subset CL$ denote the set $[0, \epsilon) \times L/\{0\} \times L$, and let $B_\epsilon(v)$ denote the ball of radius ϵ around $v \in \mathbb{R}^k$. Let $\{U_i\}$ be the collection of open subsets of X of the form $C_\epsilon L \times B_\delta(v)$ under the homeomorphism $U \cong CL \times \mathbb{R}^k$. Whenever U_i and U_j are of this form and $U_j \subset U_i$, it is easy to directly construct a stratified homotopy inverse to the inclusion map $U_j \hookrightarrow U_i$; thus by assumption the restriction functor $\mathcal{C}(U_i) \to \mathcal{C}(U_j)$ is an equivalence. For each $y \in Y \cap U$ the U_i containing y form a fundamental system of neighborhoods of y, so the functor $\mathcal{C}(U) \to \mathcal{C}_y \cong \left(j^*\mathcal{C}\right)_y$ is an equivalence. It follows that $j^*\mathcal{C}$ is equivalent to the constant sheaf on $Y \cap U$ with fiber $\mathcal{C}(U)$. This completes the proof. $\square$

3.3. **Direct images.**

Proposition 3.14. *Let (X, S) and (Y, T) be topologically stratified spaces, and let $f : X \to Y$ be a stratified map (see definition 3.4). Let $\mathcal{C}$ be an S-constructible stack on X. Then $f_*\mathcal{C}$ is T-constructible on Y.*

Proof. Let y be a point of Y. Let $U \cong \mathbb{R}^k \times CL$ be a conical neighborhood of y, and let $V \subset U$ be a smaller conical neighborhood such that the inclusion map $V \hookrightarrow U$ is a stratified homotopy equivalence. We may assume U is small enough so that there exists a topologically stratified space F and a stratum-preserving homeomorphism $f^{-1}(U) \cong F \times U$ that commutes with the projection to U. Then the inclusion map $f^{-1}(V) \hookrightarrow f^{-1}(U)$ is a stratified homotopy equivalence: if $\phi : U \to V$ is a homotopy inverse, then a homotopy inverse to $f^{-1}(V) \hookrightarrow f^{-1}(U)$ is given by $(id, \phi) : F \times U \to F \times V$. By proposition 3.12 and theorem 3.13 it follows that $f_*\mathcal{C}$ is constructible. $\square$

4. Example: the stack of perverse sheaves

Let (X,S) be a topologically stratified space. Let $D^b_S(X)$ denote the bounded constructible derived category of (X,S). $D^b_S(X)$ is the full subcategory of the bounded derived category of sheaves of abelian groups on X whose objects are the cohomologically constructible complexes of sheaves on X; that is, the complexes whose cohomology sheaves are constructible with respect to the stratification of X. For details and references see [7].

We note the following:

Lemma 4.1. *Let (X,S) be a topologically stratified space, let T be the induced stratification on $(0,1)\times X$, and let $\pi:(0,1)\times X\to X$ denote the projection map. The pullback functor $\pi^*:D^b_S(X)\to D^b_T\big((0,1)\times X\big)$ is an equivalence of categories.*

Proof. A constructible sheaf F on a topologically stratified space U of the form $U=\mathbb{R}^k\times CL$ has the property that $H^i(U;F)=0$ for $i>0$. We may use this to show that $R^i\pi_*F$ vanishes for F constructible on $(0,1)\times X$ and $i>0$. Indeed, $R^i\pi_*F$ is the sheafification of the presheaf $U\mapsto H^i\big((0,1)\times U;F)\big)$ and since every point of X has a fundamental system of neighborhoods of the form $\mathbb{R}^k\times CL$ the stalks of this presheaf vanish; it follows that $R^i\pi_*F$ vanishes. Thus $F\to\mathbf{R}\pi_*\pi^*F$ is a quasi-isomorphism for every sheaf F on X, and $\pi^*\mathbf{R}\pi_*F\to F$ is a quasi-isomorphism for every constructible sheaf on $(0,1)\times X$, completing the proof. $\square$

If C is a connected stratum of X let i_C denote the inclusion map $i_C:C\hookrightarrow X$. Let $D^b_{lc}(C)$ denote the subcategory of $D^b(C)$ whose objects are the complexes with locally constant cohomology sheaves. Recall the four functors $\mathbf{R}i_{C,*},i_{C,!}:D^b_{lc}(C)\to D^b_S(X)$ and $\mathbf{R}i^!_C,i^*_C:D^b_S(X)\to D^b_{lc}(C)$, and recall the following definition from [1]:

Definition 4.2. Let (X,S) be a topologically stratified space, and let $p:C\mapsto p(C)$ be any function from connected strata of (X,S) to $\mathbb{Z}$. For each connected stratum C, let i_C denote the inclusion $C\hookrightarrow X$. A *perverse sheaf of perversity* p on X, constructible with respect to S, is a complex $K\in D^b_S(X)$ such that

(1) The cohomology sheaves of $i^*_CK\in D^b(C)$ vanish above degree $p(C)$ for each C.
(2) The cohomology sheaves of $\mathbf{R}i^!_CK\in D^b(C)$ vanish below degree $p(C)$ for each C.

Let $\mathbf{P}(X,S,p)$ denote the full subcategory of $D^b_S(X)$ whose objects are the perverse sheaves of perversity p.

Every open set $U\subset X$ inherits a stratification from X, and we may form the category $D^b_S(U)$. This defines a prestack on X: there is a restriction functor $D^b_S(U)\to D^b_S(V)$ defined in the obvious way whenever $V\subset U$ are open sets in X. It is easy to see that if P is a perverse sheaf on U then its restriction to V is also a perverse sheaf. We obtain a prestack $U\mapsto\mathbf{P}(U,S,p)$. Write $\mathcal{P}_{X,S,p}$ for this prestack. The following theorem is a result of [1]:

Theorem 4.3. *Let X be a topologically stratified space with stratification S. Let p be any function from connected strata of X to integers. The prestack $\mathcal{P}_{X,S,p}$ is a stack.*

We may easily prove, using the criterion in theorem 3.13:

Theorem 4.4. *Let (X,S) be a topologically stratified space. Let p be any function from connected strata of X to integers. The stack $\mathcal{P}_{X,S,p}$ is constructible.*

Proof. Let U and V be open sets in X, and suppose $V \subset U$ and that the inclusion map $V \hookrightarrow U$ is a stratified homotopy equivalence. By lemma 4.1, the restriction map $D^b_S(U) \to D^b_S(V)$ is an equivalence of categories. It follows that $\mathcal{P}(U) \to \mathcal{P}(V)$ is also an equivalence. Thus $\mathcal{P}$ is constructible by theorem 3.13. □

5. The fundamental 2-groupoid and 2-monodromy

In this section we review the unstratified version of our main theorem 1.3: we introduce the fundamental 2-groupoid $\pi_{\leq 2}(X)$ of a space X and prove that the 2-category of locally constant stacks $\mathrm{St}_{lc}(X)$ is equivalent to the 2-category of **Cat**-valued functors on $\pi_{\leq 2}(X)$. Let us call the latter objects "2-monodromy functors," and write $2\mathrm{Mon}(X)$ for the 2-category of 2-monodromy functors $F : \pi_{\leq 2}(X) \to \mathbf{Cat}$. We define a 2-functor

$$\mathsf{N} : 2\mathrm{Mon}(X) \to \mathrm{St}_{lc}(X)$$

and prove that it is essentially fully faithful and essentially surjective. The most important ingredient is an analog for $\pi_{\leq 2}(X)$ of the classical van Kampen theorem; this is theorem 5.6. The results of this section are essentially contained in [9] and [18].

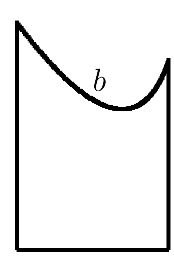
5.1. **The fundamental 2-groupoid.** Let X be a compactly generated Hausdorff space, and let x and y be two points of X. A *Moore path* from x to y is a pair (λ, γ) where λ is a nonnegative real number and $\gamma : [0, \lambda] \to X$ is a path with $\gamma(0) = x$ and $\gamma(\lambda) = y$. Let us write $P(x, y)$ for the space of Moore paths from x to y, given the compact-open topology. We have a concatenation map $P(y, z) \times P(x, y) \to P(x, z)$ defined by the formula

$$(\lambda, \gamma) \cdot (\kappa, \beta) = (\lambda + \kappa, \alpha) \text{ where } \alpha(t) = \begin{cases} \beta(t) & \text{if } t \leq \kappa \\ \gamma(t - \kappa) & \text{if } t \geq \kappa \end{cases}$$

If we give the product $P(y, z) \times P(x, y)$ the Kelly topology (the categorical product in the category of compactly generated Hausdorff spaces), this concatenation map is continuous. It is strictly associative and the constant paths from $[0, 0]$ are strict units.

Definition 5.1. Let $\pi_{\leq 2}(X)$ denote the 2-category whose objects are points of X, and whose hom categories $\mathrm{Hom}(x, y)$ are the fundamental groupoids of the spaces $P(x, y)$. (The discussion above shows that this is a strict 2-category.)

Remark 5.2. The 2-morphisms in $\pi_{\leq 2}(X)$ are technically equivalence classes of paths $[0, 1] \to P(x, y)$. A path $[0, 1] \to P(x, y)$ between α and β is given by a pair (b, H) where b is a map $[0, 1] \to \mathbb{R}_{\geq 0}$, and H is a map from the closed region in $[0, 1] \times \mathbb{R}_{\geq 0}$ under the graph of b:



H is required to take the top curve to y, the bottom curve to x, and to map the left and right intervals into X by α and β. It is inconvenient and unnecessary to keep track of the function b: there is a reparameterization map from Moore paths to ordinary (length 1) paths which takes (λ, γ) to the path $t \mapsto \gamma(\lambda \cdot t)$. This map is a homotopy equivalence, so it induces an equivalence of fundamental groupoids. Thus, 2-morphisms from α to β may be represented by homotopy classes of maps $H : [0,1] \times [0,1] \to X$ with the properties

(1) $H(0,u) = \alpha(s \cdot u)$, where s is the length of the path α.
(2) $H(1,u) = \beta(t \cdot u)$, where t is the length of the path β.
(3) $H(u,0) = x$ and $H(u,1) = y$.

5.2. **Two-monodromy and locally constant stacks.**

Definition 5.3. Let X be a compactly generated Hausdorff space. Let $2\mathrm{Mon}(X)$ denote the 2-category of 2-functors from $\pi_{\leq 2}(X)$ to the 2-category of categories:

$$2\mathrm{Mon}(X) := 2\mathrm{Funct}\big(\pi_{\leq 2}(X), \mathbf{Cat}\big)$$

Let $U \subset X$ be an open set. The inclusion morphism $U \hookrightarrow X$ induces a strict 2-functor $\pi_{\leq 2}(U) \to \pi_{\leq 2}(X)$; let j_U denote this 2-functor. If $F : \pi_{\leq 2}(X) \to \mathbf{Cat}$ is a 2-monodromy functor on X set $F|_U := F \circ j_U$.

Definition 5.4. Let X be a compactly generated Hausdorff space. Let $\mathsf{N} : 2\mathrm{Mon}(X) \to \mathrm{Prest}(X)$ denote the 2-functor which assigns to a 2-monodromy functor $F : \pi_{\leq 2}(X) \to \mathbf{Cat}$ the prestack

$$\mathsf{N}F : U \mapsto \underset{\pi_{\leq 2}(U)}{2\varprojlim} F|_U$$

Our goal is to prove that when X is locally contractible N gives an equivalence of 2-categories between $2\mathrm{Mon}(X)$ and $\mathrm{St}_{lc}(X)$; this is theorem 5.7.

5.3. **A van Kampen theorem for the fundamental 2-groupoid.** Let X be a compactly generated Hausdorff space. Let $\{U_i\}_{i \in I}$ be a d-cover of X. (By this we just mean that $\{U_i\}_{i \in I}$ is an open cover of X closed under finite intersections; then I is partially ordered by inclusion. See appendix A.) An ideal van Kampen theorem would state that the 2-category $\pi_{\leq 2}(X)$ is the direct limit (or "direct 3-limit") of the 2-categories $\pi_{\leq 2}(U_i)$. We do not wish to develop the relevant definitions here. Instead, we will relate the 2-category $\pi_{\leq 2}(X)$ to the 2-categories $\pi_{\leq 2}(U_i)$ by studying 2-monodromy functors. We will define a 2-category $2\mathrm{Mon}\big(\{U_i\}_{i \in I}\big)$ of "2-monodromy functors on the d-cover," and our van Kampen theorem will state that this 2-category is equivalent to $2\mathrm{Mon}(X)$.

If U is an open subset of X, the inclusion morphism $U \hookrightarrow X$ induces a 2-functor $\pi_{\leq 2}(U) \to \pi_{\leq 2}(X)$. Let us denote by $(-)|_U$ the 2-functor $2\mathrm{Mon}(X) \to 2\mathrm{Mon}(U)$ obtained by composing with $\pi_{\leq 2}(U) \to \pi_{\leq 2}(X)$.

Definition 5.5. Let $\{U_i\}_{i \in I}$ be a d-cover of X. A *2-monodromy functor* on $\{U_i\}_{i \in I}$ consists of the following data:

(0) For each $i \in I$, a 2-monodromy functor $F_i \in 2\mathrm{Mon}(U_i)$.
(1) For each $i, j \in I$ with $U_j \subset U_i$, an equivalence of 2-monodromy functors $F_i|_{U_j} \xrightarrow{\sim} F_j$.

(2) For each $i, j, k \in I$ with $U_k \subset U_j \subset U_i$, an isomorphism between the composite equivalence $F_i|_{U_j}|_{U_k} \xrightarrow{\sim} F_j|_{U_k} \xrightarrow{\sim} F_k$ and the equivalence $F_i|_{U_k} \xrightarrow{\sim} F_k$

such that the following condition holds:

(3) For each $i, j, k, \ell \in I$ with $U_\ell \subset U_k \subset U_j \subset U_i$, the tetrahedron commutes:

$$\begin{tikzcd} F_i|_{U_j}|_{U_k}|_{U_\ell} \arrow[r] \arrow[d] \arrow[dr] & F_j|_{U_k}|_{U_\ell} \arrow[d] \\ F_k|_{U_\ell} \arrow[r] & F_\ell \end{tikzcd} = \begin{tikzcd} F_i|_{U_j}|_{U_k}|_{U_\ell} \arrow[r] \arrow[d] & F_j|_{U_k}|_{U_\ell} \arrow[d] \arrow[dl] \\ F_k|_{U_\ell} \arrow[r] & F_\ell \end{tikzcd}$$

The 2-monodromy functors on $\{U_i\}_{i\in I}$ form the objects of a 2-category in a natural way.

If F is a 2-monodromy functor on X then we may form a 2-monodromy functor on $\{U_i\}_{i\in I}$ by setting $F_i = F|_{U_i}$, and taking all the 1-morphisms and 2-morphisms to be identities. This defines a 2-functor $2\mathrm{Mon}(X) \to 2\mathrm{Mon}(\{U_i\})$; let us denote it by *res*.

Theorem 5.6 (van Kampen)**.** *Let X be a compactly generated Hausdorff space, and let $\{U_i\}_{i\in I}$ be a d-cover of X. The natural 2-functor res : $2\mathrm{Mon}(X) \to 2\mathrm{Mon}(\{U_i\}_{i\in I})$ is an equivalence of 2-categories.*

We will prove this in section 5.5. Let us first use this result to derive our 2-monodromy theorem.

5.4. **The 2-monodromy theorem.**

Theorem 5.7. *Let X be a compactly generated Hausdorff space, and let F be a 2-monodromy functor on X. The prestack $\mathsf{N}F$ is a stack. Furthermore, if X is locally contractible, the stack $\mathsf{N}F$ is locally constant, each stalk category $(\mathsf{N}F)_x$ is naturally equivalent to $F(x)$, and the 2-functor $\mathsf{N} : 2\mathrm{Mon}(X) \to \mathrm{St}_{lc}(X)$ is an equivalence of 2-categories.*

Proof. Let G be another 2-monodromy functor on X, and let $\mathsf{N}(G, F)$ be the prestack $U \mapsto \mathrm{Hom}_{2\mathrm{Mon}(U)}(G|_U, F|_U)$. It is useful to show that $\mathsf{N}(G, F)$ is a stack; we obtain that $\mathsf{N}F = \mathsf{N}(*, F)$ is a stack as a special case.

Let $U \subset X$ be an open set, and let $\{U_i\}_{i\in I}$ be a d-cover of U. To see that the natural functor

$$\mathsf{N}(G, F)(U) \to 2\varprojlim_I \mathsf{N}(G, F)(U_i)$$

is an equivalence of categories note that $2\varprojlim_I \mathsf{N}(G, F)(U_i)$ is equivalent to the category of 1-morphisms from $res(G|_U)$ to $res(F|_U)$. Here $res(G|_U)$ and $res(F|_U)$ denote the 2-monodromy functors on the d-cover $\{U_i\}$ induced by the 2-monodromy functors $G|_U$ and $F|_U$ on U. By theorem 5.6, *res* induces an equivalence on hom categories. Thus $\mathsf{N}(G, F)$ is a stack.

Let U and V be contractible open subsets of X with $V \subset U$. Then both $\pi_{\leq 2}(V)$ and $\pi_{\leq 2}(U)$ are trivial, so $\pi_{\leq 2}(V) \to \pi_{\leq 2}(U)$ in an equivalence. It follows that $\mathsf{N}(G, F)(U) \to \mathsf{N}(G, F)(V)$ is an equivalence of categories. If X is locally contractible then by theorem 2.9 $\mathsf{N}(G, F)$ is locally constant. In fact if U is contractible and $x \in U$, the triviality of the 2-category $\pi_{\leq 2}(U)$ shows that $\mathsf{N}(G, F)(U)$ is naturally equivalent to the category of functors from $G(x)$ to $F(x)$, thus the stalk $\mathsf{N}(G, F)_x$ is equivalent to $\mathrm{Funct}(G(x), F(x))$.

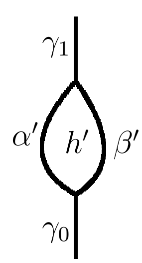
Now suppose X is locally contractible, and let us show that $\mathsf{N} : 2\mathrm{Mon}(X) \to \mathrm{St}_{lc}(X)$ is essentially fully faithful: we have to show that $\mathrm{Hom}_{2\mathrm{Mon}(X)}(G, F) \to \mathrm{Hom}_{\mathrm{St}(X)}(\mathsf{N}G, \mathsf{N}F)$ is an equivalence of categories. In fact we will show that the morphism of stacks $\mathsf{N}(G, F) \to \underline{\mathrm{Hom}}(\mathsf{N}G, \mathsf{N}F)$ is an equivalence. (Here $\underline{\mathrm{Hom}}(\mathsf{N}G, \mathsf{N}F)$ is the stack on X that takes an open set U to the category of 2-natural transformations $\underline{\mathrm{Hom}}(\mathsf{N}G(U), \mathsf{N}F(U))$.) It suffices to show that each of the functors $\mathsf{N}(G, F)_x \to \underline{\mathrm{Hom}}(\mathsf{N}G, \mathsf{N}F)_x$ between stalks is an equivalence of categories; both these categories are naturally equivalent to the category of functors $\mathrm{Funct}\big(G(x), F(x)\big)$.

Finally let us show that $\mathsf{N} : 2\mathrm{Mon}(X) \to \mathrm{St}_{lc}(X)$ is essentially surjective. For each 1-category $\mathbf{C}$, if F is the constant $\mathbf{C}$-valued 2-monodromy functor on X then $\mathsf{N}F$ is the constant stack with fiber $\mathbf{C}$: the obvious map from the constant prestack $\mathbf{C}_{p;X}$ to $\mathsf{N}F$ induces an equivalence on stalks. Thus every constant stack is in the essential image of N. Let $\mathcal{C}$ be a locally constant stack on X, and let $\{U_i\}_{i\in I}$ be a d-cover of X over which $\mathcal{C}$ trivializes. Then we may form a 2-monodromy functor on the d-cover as follows: for each $i \in I$ we may find an F_i (a constant functor) and an equivalence $\mathsf{N}F_i \cong \mathcal{C}|_{U_i}$; then for each $i, j \in I$ we may form the composite equivalence $F_i|_{U_j} \cong \mathcal{C}|_{U_i}|_{U_j} = \mathcal{C}|_{U_j} \cong F_j$; etc. By theorem 5.6, this descends to a 2-monodromy functor F on X, and $\mathsf{N}F$ is equivalent to $\mathcal{C}$. This completes the proof. □

5.5. **The proof of the van Kampen theorem.** Before proving theorem 5.6 let us discuss homotopies in more detail. We wish to show that any 2-morphism in $\pi_{\leq 2}(X)$ may be factored into smaller 2-morphisms, where "small" is interpreted in terms of an open cover of X.

Definition 5.8. Let X be a compactly generated Hausdorff space. Let $\{U_i\}_{i\in I}$ be a d-cover of X. A homotopy $h : [0, 1] \times [0, 1] \to X$ is *i-elementary* if there is a subinterval $[a, b] \subset [0, 1]$ such that $h(s, t)$ is independent of s so long as $t \notin [a, b]$, and such that the image of $[0, 1]\times[a, b] \subset [0, 1]\times[0, 1]$ under h is contained in U_i. If a homotopy h is i-elementary for some unspecified $i \in I$ then we will simply call h *elementary*.

Let X and $\{U_i\}$ be as in the definition. Let $x, y \in X$ be points, $\alpha, \beta \in P(x, y)$ be Moore paths, and let $h : [0, 1] \times [0, 1] \to X$ be a homotopy from α to β. (See remark 5.2.) Suppose we have paths $\gamma_0, \gamma_1, \alpha'$ and β', and a homotopy $h' : \alpha' \to \beta'$, such that $\alpha = \gamma_1 \cdot \alpha' \cdot \gamma_0$, $\beta = \gamma_1 \cdot \beta' \cdot \gamma_0$, and $h = 1_{\gamma_1} \cdot h' \cdot 1_{\gamma_0}$.

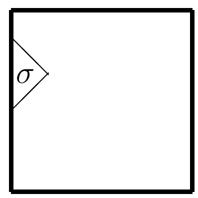


Then h is an i-elementary homotopy if and only if the image of h' lies in U_i. Any i-elementary homotopy may be written as $\gamma_1 \cdot h' \cdot \gamma_0$ for some γ_0, h', γ_1.

Proposition 5.9. *Let X be a compactly generated Hausdorff space, and let $\{U_i\}_{i\in I}$ be a d-cover of X. Let α and β be two Moore paths from x to y, and let $h : [0, 1] \times [0, 1] \to X$ be*

a homotopy from α to β. Then there is a finite list $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ of Moore paths from x to y, and of homotopies $h_1 : \alpha_0 \to \alpha_1$, $h_2 : \alpha_1 \to \alpha_2$, $\ldots h_n : \alpha_{n-1} \to \alpha_n$ such that h is homotopic to $h_n \circ h_{n-1} \circ \ldots \circ h_1$, and such that each h_i is elementary.

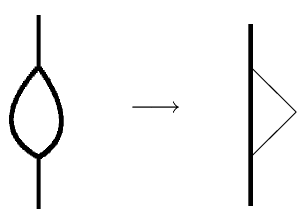
Proof. Pick a continuous triangulation of $[0,1] \times [0,1]$ with the property that each triangle is mapped by h into one of the U_i. Let n be the number of triangles, and suppose we have constructed an appropriate factorization whenever the square may be triangulated with fewer than n triangles. Pick an edge along $\{0\} \times [0,1]$; this edge is incident with a unique triangle σ, as in the diagram



We may find a homeomorphism η between the complement of σ in this square with another square such that the composition

$$[0,1] \times [0,1] \overset{\eta}{\cong} \mathrm{closure}\big([0,1] \times [0,1] - \sigma\big) \to X$$

may be triangulated with $n-1$ triangles. Let us denote this composition by g. On the other hand it is clear how to parameterize the union of σ and $\{0\} \times [0,1]$ by an elementary homotopy:



Let us write $k : [0,1] \times [0,1] \to X$ for the composition of this parameterization with h. Now k is an elementary homotopy and g may be factored into elementary homotopies by induction. The mapping cylinders on the homeomorphism η and the parameterization of $\sigma \cup \{0\} \times [0,1]$ form a homotopy between h and $g \circ k$. $\square$

We also need a notion of elementary 3-dimensional homotopy.

Definition 5.10. Let X be a compactly generated Hausdorff space, and let $\{U_i\}_{i \in I}$ be a d-cover of X. Let $x, y \in X$, $\alpha, \beta \in P(x,y)$, and let $h_0, h_1 : [0,1] \times [0,1] \to X$ be homotopies from α to β. A homotopy $t \mapsto h_t$ between h_0 and h_1 is called *i-elementary* if there is a closed rectangle $[a,b] \times [c,d] \subset [0,1] \times [0,1]$ such that

(1) $h_t(u,v)$ is independent of t for $(u,v) \notin [a,b] \times [c,d]$
(2) For each t, $h_t([a,b] \times [c,d]) \subset U_i$.

Proposition 5.11. *Let X, $\{U_i\}_{i\in I}$, x, y, α, β be as in definition 5.10. Let h and g be homotopies from α to β. Suppose that h and g are homotopic. Then there is a sequence $h = k_0, k_1, \ldots, k_n = g$ of homotopies from α to β such that k_i is homotopic to k_{i+1} via an elementary homotopy.*

Proof. Note that the homotopy between h and a factorization $h_m \circ \ldots \circ h_1$ constructed in propostion 5.9 is given by a sequence of elementary 3-dimensional homotopies. Thus we may assume that h is of the form $h_m \circ \ldots \circ h_1$, where each h_i is an elementary homotopy, and that g is of the form $g_\ell \circ \ldots \circ g_1$ where each g_i is an elementary homotopy. By induction we may reduce to the case where $m = \ell = 1$ so that h and g are both elementary. Suppose that $H : [0,1]\times[0,1]\times[0,1]$ is a homotopy between h and g. We may triangulate $[0,1]\times[0,1]\times[0,1]$ in such a way that each simplex σ is carried by H into one of the charts U_i. We may use these simplices to factor H just as in proposition 5.9. □

Now we may prove theorem 5.6. To show that $res : 2\mathrm{Mon}(X) \to 2\mathrm{Mon}\big(\{U_i\}\big)$ is an equivalence of 2-categories it suffices to show that res is essentially fully faithful and essentially surjective. This is the content of the following three propositions.

Proposition 5.12. *Let X be a compactly generated Hausdorff space, and let $\{U_i\}_{i\in I}$ be a d-cover of X. Let F and G be two 2-monodromy functors on X. The functor* $\mathrm{Hom}(F,G) \to \mathrm{Hom}\big(res(F), res(G)\big)$ *induced by res is fully faithful.*

Proof. Let n and m be two 2-natural transformations $F \to G$, and let ϕ and ψ be two modifications $n \to m$. Since we have $\phi = \psi$ if and only if $\phi_x = \psi_x$ for each $x \in X$, and since $\phi_x = res(\phi)_{x,i}$ and $\psi_x = res(\psi)_{x,i}$ whenever $x \in U_i$, we have $res(\phi) = res(\psi)$ if and only if $\phi = \psi$. This proves that the functor induced by res is faithful.

Let n and m be as before, and now let $\{\phi_{x,i}\}$ be a 2-morphism between $res(n)$ and $res(m)$. The 2-morphisms $\phi_{x,i} : n(x) \to m(x)$ are necessarily independent of i, since whenever $x \in U_j \subset U_i$ we have $\phi_{x,j} \circ 1_{n(x)} = 1_{m(x)} \circ \phi_{x,i}$. Write $\phi_x = \phi_{x,i}$ for this common value. Since $\{\phi_{x,i}\}$ is a 2-morphism in $2\mathrm{Mon}\big(\{U_i\}_{i\in I}\big)$, every path $\gamma : x \to y$ whose image is contained in one of the U_i induces a commutative diagram

$$\begin{tikzcd} n(x) \arrow[r, "n(\gamma)"] \arrow[d, "\phi_x"'] & n(y) \arrow[d, "\phi_y"] \\ m(x) \arrow[r, "m(\gamma)"'] & m(y) \end{tikzcd}$$

It follows that this diagram commutes for every path γ, since every γ may be written as a concatenation $\gamma_N \cdot \ldots \cdot \gamma_1$ of paths γ_k with the property that for each k there is an i such that the image of γ_k is contained in U_i. Thus, $x \mapsto \phi_x$ is a 2-morphism $n \to m$, and $res\big(\{\phi_x\}\big) = \{\phi_{x,i}\}$, so the functor induced by res is full. □

Proposition 5.13. *Let X, $\{U_i\}_{i\in I}$, F, and G be as in proposition 5.12. The functor* $\mathrm{Hom}(F,G) \to \mathrm{Hom}\big(res(F), res(G)\big)$ *induced by res is essentially surjective.*

Proof. Let $\{n_i\}$ be a 1-morphism $res(F) \to res(G)$. For each i and each x with $x \in U_i$ we are given a functor $n_i(x) : F(x) \to G(x)$, and for each j with $x \in U_j \subset U_i$ we are given a natural

isomorphism $\rho_{ij;x} : n_i(x) \xrightarrow{\sim} n_j(x)$ which makes certain diagrams commute. In particular, if we have $x \in U_k \subset U_j \subset U_i$, then $\rho_{jk;x} \circ \rho_{ij;x} = \rho_{ik;x}$. Let us take

$$n(x) := \varinjlim_{i \in I \,|\, U_i \ni x} n_i(x)$$

Since the limit is filtered and each $n_i(x) \to n_j(x)$ is an isomorphism, the limit exists and all the natural maps $n_i(x) \to n(x)$ are isomorphisms.

To show that $\{n_i\}$ is in the essential image of *res* we will extend the assignment $x \mapsto n(x)$ to a 1-morphism $F \to G$. To do this we need to define an isomorphism $n(\gamma) : G(\gamma) \circ n(x) \xrightarrow{\sim} n(y) \circ F(\gamma)$ for every path γ starting at x and ending at y. In case the image of γ is entirely contained in U_i for some i, define $n(\gamma)$ to be the composition

$$G(\gamma) \circ n(x) \xleftarrow{\sim} G(\gamma) \circ n_i(x) \xrightarrow{n_i(\gamma)} n_i(y) \circ F(\gamma) \xrightarrow{\sim} n(y) \circ F(\gamma)$$

By naturality of the morphisms $\rho_{ij;x}$, this map is independent of i. For general γ we may find a factorization $\gamma = \gamma_1 \cdot \ldots \cdot \gamma_N$ where each γ_k is contained in some U_ℓ, and define $n(\gamma) = n(\gamma_1) n(\gamma_2) \ldots n(\gamma_k)$.

Let x and y be points in X, let α and β be two paths from x to y, and let h be a homotopy from α to β. To show that the assignments $x \mapsto n(x)$ and $\gamma \mapsto n(\gamma)$ form a 1-morphism $F \to G$, we have to show that $n(\alpha)$ and $n(\beta)$ make the following square commute:

$$\begin{tikzcd}
n(y) \circ F(\alpha) \arrow[r, "n(y)F(h)"] \arrow[d, "n(\alpha)"'] & n(y) \circ F(\beta) \arrow[d, "n(\beta)"] \\
G(\alpha) \circ n(x) \arrow[r, "G(h)n(x)"'] & G(\beta) \circ n(x)
\end{tikzcd}$$

By proposition 5.9 we may assume h is elementary. An elementary homotopy may be factored as $h = 1_\gamma \cdot h' \cdot 1_\delta$ where the image of h' lies in U_i, so we may as well assume the image of h lies in U_i. In that case the diagram above is equivalent to

$$\begin{tikzcd}
n_i(y) \circ F|_i(\alpha) \arrow[r, "n_i(y)F|_i(h)"] \arrow[d, "n_i(\alpha)"'] & n_i(y) \circ F|_i(\beta) \arrow[d, "n_i(\beta)"] \\
G|_i(\alpha) \circ n_i(x) \arrow[r, "G|_i(h)n_i(x)"'] & G|_i(\beta) \circ n_i(x)
\end{tikzcd}$$

which commutes by assumption. (Here $F|_i$ and $G|_i$ denote the restrictions of F and G to $\pi_{\leq 2}(U_i)$.) The natural isomorphisms $n_i(x) \to n(x)$ assemble to an isomorphism between $res(n)$ and $\{n_i\}$, completing the proof. □

Proposition 5.14. *Let X be a compactly generated Hausdorff space, and let $\{U_i\}$ be a d-cover of X. The natural 2-functor $res : 2\mathrm{Mon}(X) \to 2\mathrm{Mon}(\{U_i\})$ is essentially surjective.*

Proof. Let $\{F_i\}$ be an object of $2\mathrm{Mon}(\{U_i\}_{i \in I})$. For each point $x \in X$ let $F(x)$ denote the category

$$F(x) := 2\varinjlim_{i \in I \,|\, x \in U_i} F_i(x)$$

Since I is filtered and each of the maps $F_i(x) \to F_j(x)$ is an equivalence, the natural map $F_i(x) \to F(x)$ is an equivalence of categories for each i.

We wish to extend the assignment $x \mapsto F(x)$ to a 2-functor $\pi_{\leq 2}(X) \to \mathbf{Cat}$. Let γ be a path between points x and y in X. If the image of γ is contained in some U_i then we may form $F(\gamma) : F(x) \to F(y)$ by taking the direct limit over i of the functors $c_{i,\gamma} : F(x) \to F(y)$, where $c_{i,\gamma}$ is the composition

$$F(x) \stackrel{\sim}{\leftarrow} F_i(x) \stackrel{F_i(\gamma)}{\to} F_i(y) \stackrel{\sim}{\to} F(y)$$

Whenever $U_j \subset U_i$ the natural transformation $c_{i,\gamma} \to c_{j,\gamma}$ induced by the commutative square

$$\begin{tikzcd}
F_i(x) \arrow[r, "F_i(\gamma)"] \arrow[d] & F_i(y) \arrow[d] \\
F_j(x) \arrow[r, "F_j(\gamma)"'] & F_j(y)
\end{tikzcd}$$

is an isomorphism, and the limit is filtered, so each of the maps $c_{i,\gamma} \to F(\gamma)$ is an isomorphism.

Now for each path γ not necessarily contained in one chart, pick a factorization $\gamma = \gamma_1 \cdot \ldots \cdot \gamma_N$ with the property that for each ℓ there is a k such that the image of γ_ℓ lies in U_k. If x_ℓ and $x_{\ell+1}$ denote the endpoints of γ_ℓ, let $F(\gamma) : F(x) \to F(y)$ be the functor given by the composition

$$F(x) = F(x_1) \stackrel{F(\gamma_1)}{\to} F(x_2) \stackrel{F(\gamma_2)}{\to} \ldots \stackrel{F(\gamma_N)}{\to} F(x_{N+1}) = F(y)$$

Suppose h is a homotopy between paths α and β with the property that the images of α, β, and h lie in a single chart U_i. Then define a natural transformation $F(h) : F(\alpha) \to F(\beta)$ to be the composition

$$F(\alpha) \stackrel{\sim}{\leftarrow} F_i(\alpha) \stackrel{F_i(h)}{\to} F_i(\beta) \stackrel{\sim}{\to} F(\beta)$$

If $h = 1_{\gamma_1} \cdot h' \cdot 1_{\gamma_0}$ is an elementary homotopy, such that the image of h' lies in some U_i, define $F(h) = 1_{F(\gamma_1)} \cdot F(h') \cdot 1_{F(\gamma_0)}$. If g is an arbitrary homotopy, let $g_n \circ g_{n-1} \cdots \circ g_1$ be a composition of elementary homotopies that is homotopic to g, and define $F(g) = F(g_n) \circ \cdots \circ F(g_1)$. The g_i exist by proposition 5.9, and the formula for $F(g)$ is independent of the factorization by proposition 5.11.

We may extend F to all elementary homotopies, since any elementary homotopy can be written as $1_{\gamma_1} \cdot h' \cdot 1_{\gamma_0}$ where the image of h' lies in some U_i; it follows that if h and g are elementary homotopies that are themselves homotopic by an elementary homotopy, then $F(h) = F(g)$. By propositions 5.9 and 5.11 this is well-defined.

The maps $F_i(x) \to F(x)$ assemble to a map $\{F_i\} \to \mathit{res}(F)$ in $2\mathrm{Mon}\big(\{U_i\}\big)$. As each $F_i(x) \to F(x)$ is an equivalence by construction, this shows that $\{F_i\}$ is equivalent to $\mathit{res}(F)$, so that res is essentially surjective. $\square$

6. Stratified 2-truncations and 2-monodromy

In this section we develop an abstract version of our main theorem. We introduce the notion of a *stratified 2-truncation*. A stratified 2-truncation $\overrightarrow{\pi}_{\leq 2}$ is a strict functorial assignment from topologically stratified spaces to 2-categories satisfying a few axioms. We show that these axioms guarantee that the 2-category of 2-functors from $\overrightarrow{\pi}_{\leq 2}(X, S)$ to **Cat** is equivalent to the 2-category of S-constructible stacks on X.

Let **Strat** denote the category of topologically stratified spaces and stratum-preserving maps between them. We will consider functors from **Strat** to the category (that is, 1-category) of 2-categories and strict 2-functors; we will denote the latter category by **2cat**. Thus, such a functor $\overrightarrow{\pi}_{\leq 2}$ consists of

1. an assignment $(X, S) \mapsto \overrightarrow{\pi}_{\leq 2}(X, S)$ that takes a topologically stratified space to a 2-category.
2. an assignment $f \mapsto \overrightarrow{\pi}_{\leq 2}(f)$ that takes a stratum-preserving map $f : X \to Y$ to a strict 2-functor $\overrightarrow{\pi}_{\leq 2}(f) : \overrightarrow{\pi}_{\leq 2}(X) \to \overrightarrow{\pi}_{\leq 2}(Y)$.

such that for any pair of composable stratum-preserving maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have $\overrightarrow{\pi}_{\leq 2}(g \circ f) = \overrightarrow{\pi}_{\leq 2}(g) \circ \overrightarrow{\pi}_{\leq 2}(f)$.

A functor $\overrightarrow{\pi}_{\leq 2}$ is called a *stratified 2-truncation* if it satisfies the four axioms below. Two of these axioms require some more discussion, but we will state them here first somewhat imprecisely:

Definition 6.1. Let $\overrightarrow{\pi}_{\leq 2}$ be a functor **Strat** $\to$ **2cat**. We will say that $\overrightarrow{\pi}_{\leq 2}$ is a *stratified 2-truncation* if it satisfies the following axioms:

- (N) Normalization. If $\emptyset$ denotes the empty topologically stratified space, then $\overrightarrow{\pi}_{\leq 2}(\emptyset)$ is the empty 2-category.
- (H) Homotopy invariance. For each topologically stratified space (X, S), the 2-functor $\overrightarrow{\pi}_{\leq 2}(f) : \overrightarrow{\pi}\big((0,1) \times X, S'\big) \to \overrightarrow{\pi}(X, S)$ induced by the projection map $f : (0,1) \times X \to X$ is an equivalence of 2-categories. Here S' denotes the stratification on $(0,1) \times X$ induced by S.
- (C) Cones. Roughly, for each compact topologically stratified space L, $\overrightarrow{\pi}_{\leq 2}(CL)$ may be identified with the cone on the 2-category $\overrightarrow{\pi}_{\leq 2}(L)$. See section 6.1 below.
- (vK) van Kampen. Roughly, for every topologically stratified space X and every d-cover $\{U_i\}_{i \in I}$ of X, the 2-category $\overrightarrow{\pi}_{\leq 2}(X)$ is naturally equivalent to the direct limit (or "3-limit") over $i \in I$ of the 2-categories $\overrightarrow{\pi}_{\leq 2}(U_i)$. See section 6.2 below.

6.1. **Cones on 2-categories and axiom (C).** If $\mathbf{C}$ is a 2-category, let $\big(* \downarrow \mathbf{C}\big)$ denote the 2-category whose objects are the objects of $\mathbf{C}$ together with one new object $*$, and where the hom categories $\mathrm{Hom}_{*\downarrow\mathbf{C}}(x, y)$ are as follows:

1. $\mathrm{Hom}(x, y) = \mathrm{Hom}_{\mathbf{C}}(x, y)$ if both x and y are in $\mathbf{C}$.
2. $\mathrm{Hom}(x, y)$ is the trivial category if $x = *$.
3. $\mathrm{Hom}(x, y)$ is the empty category if $y = *$ and $x \neq *$.

Definition 6.2. Let $\overrightarrow{\pi}_{\leq 2}$: **Strat** $\to$ **2cat** be a functor satisfying axioms (N) and (H) above. For each compact topologically stratified space L, let us endow $(0,1) \times L$ and CL with the naturally induced topological stratification. Let us say that $\overrightarrow{\pi}_{\leq 2}$ *satisfies axiom*

(C) if for each compact topologically stratified space L there is an equivalence of 2-categories $\overrightarrow{\pi}_{\leq 2}(CL) \xrightarrow{\sim} \big(* \downarrow \overrightarrow{\pi}_{\leq 2}(L)\big)$ such that

(1) the following square commutes up to equivalence of 2-functors:

$$\begin{tikzcd}
\overrightarrow{\pi}_{\leq 2}\big((0,1) \times L\big) \arrow[r] \arrow[d] & \overrightarrow{\pi}_{\leq 2}(CL) \arrow[d] \\
\overrightarrow{\pi}_{\leq 2}(L) \arrow[r] & \big(* \downarrow \overrightarrow{\pi}_{\leq 2}(L)\big)
\end{tikzcd}$$

(2) The composition $\overrightarrow{\pi}_{\leq 2}\big(\{\text{cone point}\}\big) \to \overrightarrow{\pi}_{\leq 2}(CL) \to \big(* \downarrow \overrightarrow{\pi}_{\leq 2}(L)\big)$ is equivalent to the natural inclusion $\overrightarrow{\pi}_{\leq 2}\big(\{\text{cone point}\}\big) \cong * \to \big(* \downarrow \overrightarrow{\pi}_{\leq 2}(L)\big)$

6.2. **Exit 2-monodromy functors and axiom (vK).** Morally, the van Kampen axiom states that $\overrightarrow{\pi}_{\leq 2}$ preserves direct limits (at least in a diagram of open immersions). We find it inconvenient to define a direct 3-limit of 2-categories directly; we will instead formulate it in terms of category-valued 2-functors on the 2-categores $\overrightarrow{\pi}_{\leq 2}(X)$, as in section 5.3.

In this section, fix a functor $\overrightarrow{\pi}_{\leq 2} : \textbf{Strat} \to \textbf{2cat}$.

Definition 6.3. Let (X, S) be a topologically stratified space. An *exit 2-monodromy functor* on (X, S) with respect to $\overrightarrow{\pi}_{\leq 2}$ is a 2-functor $\overrightarrow{\pi}_{\leq 2}(X, S) \to \textbf{Cat}$. Write $2\mathrm{Exitm}(X, S) = 2\mathrm{Exitm}(X, S; \overrightarrow{\pi}_{\leq 2})$ for the 2-category of exit 2-monodromy functors on (X, S) with respect to $\overrightarrow{\pi}_{\leq 2}$.

Definition 6.4. Let (X, S) be a topologically stratified space. Let $\{U_i\}_{i \in I}$ be a d-cover of X. Endow each U_i with the topological stratification S_i inherited from S. An *exit 2-monodromy functor* on $\{U_i\}_{i \in I}$, with respect to $\overrightarrow{\pi}_{\leq 2}$ consists of the following data:

(0) For each $i \in I$ a 2-monodromy functor $F_i \in 2\mathrm{Exitm}(U_i, \overrightarrow{\pi}_{\leq 2})$.

(1) For each $i, j \in I$ with $U_j \subset U_i$ an equivalence of exit 2-monodromy functors $F_i|_{U_j} \xrightarrow{\sim} F_j$.

(2) For each $i, j, k \in I$ with $U_k \subset U_j \subset U_i$ an isomorphism between the composite equivalence $F_i|_{U_j}|_{U_k} \xrightarrow{\sim} F_j|_{U_k} \xrightarrow{\sim} F_k$ and the equivalence $F_i|_{U_k} \xrightarrow{\sim} F_k$

such that the following condition holds:

(3) For each $i, j, k, \ell \in I$ with $U_\ell \subset U_k \subset U_j \subset U_i$, the tetrahedron commutes:

$$\begin{tikzcd}
F_i|_{U_j}|_{U_k}|_{U_\ell} \arrow[r] \arrow[d] \arrow[dr] & F_j|_{U_k}|_{U_\ell} \arrow[d] \\
F_k|_{U_\ell} \arrow[r] & F_\ell
\end{tikzcd}
=
\begin{tikzcd}
F_i|_{U_j}|_{U_k}|_{U_\ell} \arrow[r] \arrow[d] & F_j|_{U_k}|_{U_\ell} \arrow[d] \arrow[dl] \\
F_k|_{U_\ell} \arrow[r] & F_\ell
\end{tikzcd}$$

Write $2\mathrm{Exitm}\big(\{U_i\}_{i \in I}, \overrightarrow{\pi}_{\leq 2}\big)$ for the 2-category of exit 2-monodromy functors on $\{U_i\}$.

Let X be a topologically stratified space, and let $\{U_i\}_{i \in I}$ be a d-cover of X. Denote by *res* the natural strict 2-functor $2\mathrm{Exitm}(X) \to 2\mathrm{Exitm}\big(\{U_i\}_{i \in I}\big)$.

Definition 6.5. Let $\overrightarrow{\pi}_{\leq 2}$ be a 2-functor $\textbf{Strat} \to \textbf{2cat}$. We say that $\overrightarrow{\pi}_{\leq 2}(X)$ *satisfies axiom (vK)* if *res* : $2\mathrm{Exitm}(X) \to 2\mathrm{Exitm}\big(\{U_i\}_{i \in I}\big)$ is an equivalence of 2-categories for every topologically stratified space X and every d-cover $\{U_i\}_{i \in I}$ of X.

6.3. **The exit 2-monodromy theorem.** In this section fix a stratified 2-truncation $\overrightarrow{\pi}_{\leq 2}$.

Definition 6.6. Let (X, S) be a topologically stratified space. For each open set $U \subset X$, let S_U denote the induced stratification of U and let j_U denote the inclusion map $U \hookrightarrow X$. Let $\mathsf{N} : 2\mathrm{Exitm}(X, S) \to \mathrm{Prest}(X)$ denote the 2-functor which assigns to an exit 2-monodromy functor $F : \overrightarrow{\pi}_{\leq 2}(X, S) \to \mathbf{Cat}$ the prestack

$$\mathsf{N}F : U \mapsto \underset{\overrightarrow{\pi}_{\leq 2}(U, S_U)}{2\varprojlim} F \circ \overrightarrow{\pi}_{\leq 2}(j_U)$$

We wish to prove that N is an equivalence of $2\mathrm{Exitm}(X, S)$ onto the 2-category $\mathrm{St}_S(X)$.

We need a preliminary result about constructible stacks on cones.

Definition 6.7. Let (L, S) be a topologically stratified space. Let $\big(\mathbf{Cat} \downarrow \mathrm{St}_S(L)\big)$ denote the 2-category whose objects are triples $(\mathbf{C}, \mathcal{C}, \phi)$, where

(1) $\mathbf{C}$ is a 1-category.
(2) $\mathcal{C}$ is a constructible stack on L
(3) ϕ is a 1-morphism $\mathbf{C}_L \to \mathcal{C}$, where $\mathbf{C}_L$ denotes the constant stack on L.

If (L, S) is a compact topologically stratified space, let S' denote the induced stratification on $(0, 1) \times L$ and S'' the induced stratification on CL. There is a 2-functor

$$\mathrm{St}_{S''}(CL) \to (\mathbf{Cat} \downarrow \mathrm{St}_{S'}((0,1) \times L)) \cong (\mathbf{Cat} \downarrow \mathrm{St}_S(L))$$

which associates to a stack $\mathcal{C}$ the triple $\big(\mathcal{C}(X), \mathcal{C}|_{(0,1)\times L}, \phi\big)$, where ϕ is the evident restriction map.

Definition 6.8. Let (L, S) be a topologically stratified space. Let $\big(\mathbf{Cat} \downarrow 2\mathrm{Exitm}(L)\big)$ denote the 2-category whose objects are triples $(\mathbf{C}, F, \phi)$ where

(1) $\mathbf{C}$ is a 1-category.
(2) F is an exit 2-monodromy functor on L.
(3) ϕ is a 1-morphism from the constant $\mathbf{C}$-valued functor to F.

Note that the equivalence

$$\overrightarrow{\pi}_{\leq 2}(CL) \xrightarrow{\sim} \big(* \downarrow \overrightarrow{\pi}_{\leq 2}(L)\big)$$

gives an equivalence

$$(\mathbf{Cat} \downarrow 2\mathrm{Exitm}(L)) \xrightarrow{\sim} 2\mathrm{Exitm}(CL)$$

.

Proposition 6.9. *Let L be a compact topologically stratified space, and let CL be the open cone on L. The 2-functor $\mathrm{St}_{S''}(CL) \to \big(\mathbf{Cat} \downarrow \mathrm{St}_S(L)\big)$ is an equivalence of 2-categories. Furthermore, the square*

$$\begin{tikzcd} 2\mathrm{Exitm}(CL, S'') \arrow[r, "\mathsf{N}"] \arrow[d] & \mathrm{St}_{S''}(CL) \arrow[d] \\ \big(\mathbf{Cat} \downarrow 2\mathrm{Exitm}(L, S)\big) \arrow[r, "\mathsf{N}"'] & \big(\mathbf{Cat} \downarrow \mathrm{St}_S(L)\big) \end{tikzcd}$$

commutes up to an equivalence of 2-functors.

Proof. The 2-functor $\mathrm{St}_{S''}(CL) \to \big(\mathbf{Cat} \downarrow \mathrm{St}_{S'}\big((0,1) \times L\big)\big)$ is inverse to the 2-functor that takes an object $(\mathbf{C}, \mathcal{C}, \phi)$ to the unique stack given by the formula

$$U \mapsto \begin{cases} \mathbf{C} & \text{if } U \text{ is of the form } C_\epsilon L = [0,\epsilon) \times L/\{0\} \times L \\ \mathcal{C}(U) & \text{if } U \text{ does not contain the cone point} \end{cases}$$

□

Theorem 6.10. *Let (X,S) be a topologically stratified space, and let F be an exit 2-monodromy functor on (X,S). The prestack $\mathsf{N}F$ is an S-constructible stack, and the 2-functor $\mathsf{N} : \mathrm{2Exitm}(X,S) \to \mathrm{St}_S(X)$ is an equivalence of 2-categories.*

Proof. We will follow the proof of theorem 5.7. Let G be another exit 2-monodromy functor on X, and once again let $\mathsf{N}(G,F)$ be the prestack $U \mapsto \mathrm{Hom}_{\mathrm{2Exitm}(U)}(G|_U, F|_U)$. As in the proof of theorem 5.7, the van Kampen property of $\overrightarrow{\pi}_{\leq 2}$ (axiom (vK)) implies $\mathsf{N}(G,F)$ is a stack. By the homotopy axiom (H), $\overrightarrow{\pi}_{\leq 2}(V) \to \overrightarrow{\pi}_{\leq 2}(U)$ is an equivalence of 2-categories whenever $V \subset U$ are open sets and $V \hookrightarrow U$ is a loose stratified homotopy equivalence. It follows that the stacks $\mathsf{N}(G,F)$ are constructible by theorem 3.13. In particular $\mathsf{N}F$ is a constructible stack.

To see that $\mathsf{N} : \mathrm{2Exitm}(X,S) \to \mathrm{St}_S(X)$ is essentially fully faithful, it suffices to show that $\mathsf{N}(G,F) \to \underline{\mathrm{Hom}}(\mathsf{N}G, \mathsf{N}F)$ is an equivalence of stacks, and we may check this on stalks. We will induct on the dimension of X: it is clear that this morphism is an equivalence of stacks when X is 0-dimensional, so suppose we have proven it an equivalence for X of dimension $\leq d$. Let $x \in X$ and let U be a conical neighborhood of x. The morphism $\mathsf{N}(G,F)_x \to \underline{\mathrm{Hom}}(\mathsf{N}G, \mathsf{N}F)_x$ is equivalent to the morphism $\mathsf{N}(G,F)(U) \to \mathrm{Hom}(\mathsf{N}G|_U, \mathsf{N}F|_U)$, and by the stratified homotopy equivalence $U \simeq CL$ we may as well assume $U = CL$. Let T denote the stratification on L. By proposition 6.9, we have to show that the 2-functor $\big(\mathbf{Cat} \downarrow \mathrm{2Exitm}(L,T)\big) \to \big(\mathbf{Cat} \downarrow \mathrm{St}_T(L)\big)$ is an equivalence, but this map is induced by $\mathsf{N} : \mathrm{2Exitm}(L,T) \to \mathrm{St}_T(L)$ which is an equivalence by induction.

Finally let us show that $\mathsf{N} : \mathrm{2Exitm}(X,S) \to \mathrm{St}_S(X)$ is essentially surjective. Again let us induct on the dimension of X. Let $\mathcal{C}$ be a constructible stack on X. The restriction of $\mathcal{C}$ to a conical open set $U \cong \mathbb{R}^d \times CL$ is in the essential image of $\mathsf{N} : \mathrm{2Exitm}(U, S_U) \to \mathrm{St}_{S_U}(U)$ by induction and proposition 6.9. We may find a d-cover $\{U_i\}_{i \in I}$ of X generated by conical open sets, so that for each i there is an $F_i \in \mathrm{2Exitm}(U_i)$ such that $\mathcal{C}|_{U_i}$ is equivalent to $\mathsf{N}F_i$. These F_i assemble to an exit 2-monodromy functor on the d-cover, which by axiom (vK) comes from an exit 2-monodromy functor F on X with $\mathsf{N}F \cong \mathcal{C}$.

This completes the proof.

□

7. Exit paths in a stratified space

In this section we identify a particular stratified 2-truncation: the exit-path 2-category $EP_{\leq 2}$. If (X,S) is a topologically stratified space, then the objects of $EP_{\leq 2}(X,S)$ are the points of X, the morphisms are Moore paths with the "exit property" described in the introduction, and the 2-morphisms are homotopy classes of homotopies between exit paths, subject to a tameness condition. The purpose of this section is to give a precise definition of the functor $EP_{\leq 2}$, and to check the axioms 6.1.

Definition 7.1. Let X be a topologically stratified space. A path $\gamma : [a, b] \to X$ is called an *exit path* if for each $t_1, t_2 \in [a, b]$ with $t_1 \leq t_2$, the point $\gamma(t_1)$ is in the closure of the stratum containing $\gamma(t_2)$; equivalently, if the dimension of the stratum containing $\gamma(t_1)$ is not larger than the dimension of the stratum containing $\gamma(t_2)$. For each pair of points $x, y \in X$ let $EP(x, y)$ denote the subspace of the space $P(x, y)$ of Moore paths (section 5.1) with the exit property, starting at x and ending at y.

Remark 7.2. If we wish to emphasize the space X we will sometimes write $EP(X; x, y)$ for $EP(x, y)$.

7.1. **Tame homotopies.** Let (X, S) be a topologically stratified space. Let us call a map $[0, 1]^n \to X$ *tame* with respect to S if there is a continuous triangulation of $[0, 1]^n$ such that the interior of every simplex maps into a stratum of X. Note that the composition of a tame map $[0, 1]^n \to X$ with a stratum-preserving map $(X, S) \to (Y, T)$ is again tame.

If x and y are two points of X, call a path $h : [0, 1] \to EP(x, y)$ tame if the associated homotopy $[0, 1]\times[0, 1] \to X$ is tame with respect to S. (See remark 5.2 for how to associate an ordinary "square" homotopy to a homotopy between Moore paths.) Finally if $H : [0, 1]\times[0, 1]$ is a homotopy between paths h and g in $EP(x, y)$, we call H tame if the associated map $[0, 1] \times [0, 1] \times [0, 1] \to X$ is tame with respect to S.

Definition 7.3. Let (X, S) be a topologically stratified space, and let x and y be points of X. Let $tame(x, y)$ be the groupoid whose objects are the points of $EP(x, y)$ and whose hom sets $\mathrm{Hom}_{tame(x,y)}(\alpha, \beta)$ are tame homotopy classes of tame paths $h : [0, 1] \to EP(x, y)$ starting at α and ending at β.

The concatenation map $EP(y, z) \times EP(x, y) \to EP(x, z)$ takes a pair of tame homotopies $h : [0, 1] \to EP(x, y)$ and $k : [0, 1] \to EP(y, z)$ to a tame homotopy $k \cdot h : [0, 1] \to EP(x, z)$, and this gives a well-defined functor $tame(y, z) \times tame(x, y) \to tame(x, z)$. It follows we may define a 2-category:

Definition 7.4. Let (X, S) be a topologically stratified space. Let $EP_{\leq 2}(X, S)$ denote the 2-category whose objects are points of X and whose hom categories $\mathrm{Hom}_{EP_{\leq 2}(X,S)}(x, y)$ are the groupoids $tame(x, y)$.

Remark 7.5. The tameness condition is necessary for our proof of the van Kampen property of $EP_{\leq 2}(X, S)$ (which follows the proof given in section 5.5) – it allows us to subdivide our homotopies indefinitely. We can define a similar 2-category $EP^{\mathrm{naive}}_{\leq 2}(X, S)$ whose hom categories are the fundamental groupoids $\pi_{\leq 1}\big(EP(x, y)\big)$. I *believe* this 2-category to be naturally equivalent to $EP_{\leq 2}(X, S)$, and that $EP^{\mathrm{naive}}_{\leq 2}$ could be used in place of $EP_{\leq 2}$ in our main theorem. To prove this one would have to show that the natural functor $tame(x, y) \to \pi_{\leq 1}\big(EP(x, y)\big)$ is an equivalence of groupoids. I have been unable to obtain such a "tame approximation" result.

7.2. **The exit path 2-category is a stratified 2-truncation.** As a stratum-preserving map $f : (X, S) \to (Y, T)$ preserves tameness of maps $[0, 1]^n \to X$, it induces a functor $f_* : tame(x, y) \to tame\big(f(x), f(y)\big)$ and a strict 2-functor $f_* : EP_{\leq 2}(X, S) \to EP_{\leq 2}(Y, T)$. Thus, $EP_{\leq 2}$ is a functor **Strat** $\to$ **2cat**. The remainder of this section is devoted to showing that $EP_{\leq 2}$ satisfies the axioms 6.1 for a stratified 2-truncation.

Theorem 7.6. *The 2-functor $EP_{\leq 2}$: **Strat** $\to$ **2cat** satisfies axioms (N) and (H) of 6.1.*

Proof. Clearly $EP_{\leq 2}(\emptyset)$ is empty, so $EP_{\leq 2}$ satisfies axiom (N).

Let us now verify axiom (H). Let (X, S) be a topologically stratified space, and let $\pi : (0,1) \times X \to X$ denote the projection map. The 2-functor $\pi_* : EP_{\leq 2}\big((0,1) \times X\big) \to EP_{\leq 2}(X)$ is clearly essentially surjective. Let (s, x) and (t, y) be two points in $(0,1) \times X$. To show that π_* is essentially fully faithful we have to show that $tame\big((s,x),(t,y)\big) \to tame(x,y)$ is an equivalence of groupoids. In fact this map is equivalent to the projection $tame(s,t) \times tame(x,y) \to tame(x,y)$, and the groupoid $tame(s,t)$ coincides with the fundamental groupoid $\pi_{\leq 1}\big(EP(s,t)\big)$ as $(0,1)$ has a single stratum. Since $EP(s,t)$ is contractible, this groupoid is equivalent to the trivial groupoid, so the projection $tame(s,t) \times tame(x,y) \to tame(x,y)$ is an equivalence.

□

Theorem 7.7. *The 2-functor $EP_{\leq 2}$: **Strat** $\to$ **2cat** satisfies axiom (C) of 6.1*

Proof. Let (L, S) be a compact topologically stratified space. Let CL be the open cone on L, and let $* \in CL$ be the cone point. We have to show that for each $x \in CL$, the groupoid $tame(*, x)$ is equivalent to the trivial groupoid. This is clear when x is the cone point, so suppose $x = (u, y) \in (0,1) \times L \subset CL$. Let $tame' \subset tame(*, x)$ denote the full subgroupoid whose objects are the exit paths α of Moore length 1 (i.e. $\alpha : [0,1] \to CL$) with $\alpha(t) \neq *$ for $t > 0$. Every exit path $\gamma \in tame(x,y)$ is clearly tamely homotopic to one in $tame'$; it follows that $tame'$ is equivalent $tame(*, x)$.

Let $W \subset EP(*, x)$ be the subspace of exit paths α with Moore length 1 and with $\alpha(t) \neq *$ for $t > 0$. W is homeomorphic to the space of paths $\beta : (0,1] \to (0,1) \times L$ with the property that β is an exit path, that $\beta(1) = x$, and that for all $\epsilon > 0$, there is a $\delta > 0$ such that $\beta^{-1}\big((0,\epsilon) \times L\big) \supset (0, \delta)$. This space may be expressed as a product $W \cong W_1 \times W_2$, where

(1) W_1 is the space of paths $\alpha : (0,1] \to (0,1)$ with $\alpha(1) = u$ and $\forall \epsilon \exists \delta$ such that $\alpha^{-1}(0,\epsilon) \supset (0,\delta)$

(2) W_2 is the space of paths $\beta : (0,1] \to L$ with $\beta(1) = y$ and β has the exit property.

The first factor W_1 is contractible via $\kappa_t : W_1 \to W_1$, where $\kappa_t(\alpha)(s) = t \cdot s \cdot u + (1-t) \cdot \alpha(s)$. The second factor is contractible via $\mu_t : W_2 \to W_2$ where $\mu_t(\beta)(s) = \beta(t + s - ts)$. These contractions preserve tameness, and therefore they induce an equivalence between $tame'$ and the trivial groupoid. □

Finally we have to prove that $EP_{\leq 2}$ satisfies the van Kampen axiom. Let us first discuss elementary tame homotopies, analogous to the elementary homotopies used in the proof of the van Kampen theorem for $\pi_{\leq 2}$ in section 5.5.

Definition 7.8. Let (X, S) be a topologically stratified space, and let $\{U_i\}_{i \in I}$ be a d-cover of X. Let x and y be points of X, and let α and β be exit paths from x to y. A homotopy $h : [0,1] \times [0,1] \to X$ between α and β is *i-elementary* if there is a subinterval $[a,b] \subset [0,1]$ such that $h(s,t)$ is independent of s so long as $t \notin [a,b]$, and such that the image of $[0,1] \times [a,b] \subset [0,1] \times [0,1]$ under h is contained in U_i.

Remark 7.9. Elementary homotopies between exit paths may be pictured in the same way as ordinary homotopies, as in figure 5.5.

Proposition 7.10. *Let (X, S) be a topologically stratified space, and let $\{U_i\}_{i\in I}$ be a d-cover of X. Let α and β be exit paths from x to y, and let $h : [0,1] \times [0,1] \to X$ be a homotopy from α to β. Then there is a finite list $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ of exit paths from x to y, and of homotopies $h_1 : \alpha_0 \to \alpha_1$, $h_2 : \alpha_1 \to \alpha_2$, $\ldots$, $h_n : \alpha_{n-1} \to \alpha_n$ such that h is homotopic to $h_n \circ \ldots \circ h_1$, and such that each h_i is elementary.*

Proof. As in the proof of proposition 5.9, it suffices to find a suitable triangulation of $[0,1] \times [0,1]$. In our case a triangulation is "suitable" if each triangle is mapped into one of the charts U_i, and if furthermore for each triangle σ we may order the vertices v_1, v_2, v_3 in such a way that h carries the half-open line segment $\overline{v_1 v_2} - v_1$ into a stratum X_k, and the third-open triangle $\overline{v_1 v_2 v_3} - \overline{v_1 v_2}$ into a stratum X_ℓ. In that case we may find a parameterization $g : [0,1] \times [0,1] \to \sigma$ of σ with the property that for each t the path $[0,1] \to \{t\} \times [0,1] \to \sigma \to X$ has the exit property in X. We may find a triangulation with these properties by picking a triangulation that is fine enough with respect to $\{U_i\}$, and taking its barycentric subdivision. ☐

We also may discuss elementary 3-dimensional homotopies between homotopies between exit paths, and a version of proposition 5.11 holds.

Definition 7.11. Let (X, S) be a topologically stratified space, and let $\{U_i\}_{i\in I}$ be a d-cover of X. Let $x, y \in X$, $\alpha, \beta \in EP^M(x, y)$, and let $h_0, h_1 : [0,1] \times [0,1] \to X$ be homotopies from α to β. A homotopy $t \mapsto h_t$ between h_0 and h_1 is called *i-elementary* if there is a closed rectangle $[a, b] \times [c, d] \subset [0,1] \times [0,1]$ such that

(1) $h_t(u, v)$ is independent of t for $(u, v) \notin [a, b] \times [c, d]$.
(2) For each t, $h_t\big([a, b] \times [c, d]\big) \subset U_i$.

Proposition 7.12. *Let (X, S), $\{U_i\}$, x, y, α, β be as in definition 7.11. Let h and g be homotopies from α to β. Suppose that h and g are homotopic. Then there is a sequence $h = k_0, k_1, \ldots, k_n = g$ of homotopies from α to β such that, for each i, k_i is homotopic to k_{i+1} via an elementary homotopy.*

Proof. Similar to proposition 5.11. ☐

Theorem 7.13. *The functor $EP_{\leq 2}$: **Strat** $\to$ **2cat** satisfies axiom (vK).*

Proof. Let X be a topologically stratified space, and let $\{U_i\}_{i\in I}$ be a d-cover of X. We have to show that res : 2Exitm$(X) \to$ 2Exitm$\big(\{U_i\}\big)$ is an equivalence of 2-categories. It suffices to show that res is essentially fully faithful and essentially surjective. The proofs of these facts in the unstratified case – propositions 5.12, 5.13, and 5.14 – may be followed almost verbatim to obtain the same results, after substituting propositions 7.10 and 7.12 for propositions 5.9 and 5.11. ☐

7.3. **Proof of the main theorem.** We may now prove the main theorem stated in the introduction.

Theorem 7.14. *The 2-functor $EP_{\leq 2}$ is a stratified 2-truncation. Because of this, for any topologically stratified space (X, S) the 2-category of S-constructible stacks on X is naturally equivalent to the 2-category of* **Cat***-valued 2-functors on $EP_{\leq 2}(X, S)$.*

Proof. That $EP_{\leq 2}$ satisfies the axioms of a stratified 2-truncation is the content of theorems 7.6, 7.7, and 7.13. The conclusion that the main theorem holds is implied by theorem 6.10. □

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Appendix A. Stacks

The word "stack" has at at least two different and related meanings in mathematics. Maybe most frequently it refers to some kind of geometric object that represents a groupoid-valued, rather than a set-valued, functor. But it may also refer to a sheaf of categories, where the sheaf structure and axioms have been modified to take account of the "two-dimensional" nature of categories – that is, to take account of the fact that categories are most naturally viewed as the objects of a 2-category. In this appendix we develop some basic properties of stacks in the second sense.

We will proceed in a way that emphasizes the similarity with sheaves. It requires generalizing the basic definitions of category theory, such as limits and adjoint functors, to 2-categories. We summarize what we need from the theory of 2-categories in appendix B.

A.1. **Prestacks.**

Definition A.1. A *prestack* on a 2-category $\mathbf{I}$ is a 2-functor $\mathcal{C} : \mathbf{I}^{op} \to \mathbf{Cat}$. A prestack on a topological space X is a prestack on the partially ordered set of open subsets of X, regarded as a 2-category whose objects are the open sets, whose 1-morphisms are the inclusion maps, and with trivial 2-morphisms. In detail, a prestack $\mathcal{C}$ on a space X consists of:

(0) An assignment that takes an open set U to a category $\mathcal{C}(U)$.
(1) A contravariant assignment that takes an inclusion $V \subset U$ to a restriction functor $\mathcal{C}(U) \to \mathcal{C}(V)$
(2) For each triple of open subsets $W \subset V \subset U$ an isomorphism between $\mathcal{C}(U) \to \mathcal{C}(V) \to \mathcal{C}(W)$ and $\mathcal{C}(U) \to \mathcal{C}(W)$
(3) Such that the tetrahedron associated to each quadruple $Y \subset W \subset V \subset U$ commutes:

$$\begin{tikzcd} \mathcal{C}(U) \arrow[r] \arrow[d] \arrow[dr] & \mathcal{C}(V) \arrow[d] \\ \mathcal{C}(W) \arrow[r] & \mathcal{C}(Y) \end{tikzcd} = \begin{tikzcd} \mathcal{C}(U) \arrow[r] \arrow[d] & \mathcal{C}(V) \arrow[d] \arrow[dl] \\ \mathcal{C}(W) \arrow[r] & \mathcal{C}(Y) \end{tikzcd}$$

Write $\mathrm{Prest}(\mathbf{I})$ for the 2-category of prestacks on a 2-category $\mathbf{I}$, and $\mathrm{Prest}(X)$ for the 2-category of prestacks on a space X.

Definition A.2. Let $\mathcal{C}$ be a prestack on X. The *stalk* of $\mathcal{C}$ at $x \in X$ is the category

$$\mathcal{C}_x =_{def} 2\varinjlim_{U \mid U \ni x} \mathcal{C}(U)$$

Remark A.3. Let $\mathbf{I}$ be a 1-category. The 2-category of 2-functors $\mathbf{I}^{op} \to \mathbf{Cat}$ is equivalent to the 2-category of so-called *fibered categories* over $\mathbf{I}$ ([10]). The theory of stacks is usually ([10], [6], [24]) developed using fibered categories rather than 2-functors.

A.2. **Stacks.** A *stack* is a prestack on X that satisfies a kind of sheaf condition. We find it convenient to phrase this condition in terms of 2-limits over an open cover; in order to make this precise we need our open covers to be closed under finite intersections. We will call these "descent covers" or d-covers.

Definition A.4. A *d-cover* of a space U is a subset $I \subset \mathrm{Open}(U)$ of the set of open subsets of U that is closed under finite intersections, and that covers U.

Let $\mathcal{C}$ be a prestack on X. If U is an open subset of X and $\{U_i\}_{i\in I}$ is a d-cover of U, then the restriction functors $\mathcal{C}(U) \to \mathcal{C}(U_i)$ assemble to a functor

$$\mathcal{C}(U) \to 2\varprojlim_I \mathcal{C}(U_i)$$

Definition A.5. Let $\mathcal{C}$ be a prestack on a space X. Then $\mathcal{C}$ is a *stack* if for each open set $U \subset X$ and each d-cover $\{U_i\}_{i\in I}$ of U, the restriction functor

$$\mathcal{C}(U) \to 2\varprojlim_I \mathcal{C}(U_i)$$

is an equivalence of categories. Let $\mathrm{St}(X) \subset \mathrm{Prest}(X)$ denote the full subcategory of the 2-category of prestacks on X whose objects are stacks.

Theorem A.6.
1. *Let $\mathcal{P}$ and $\mathcal{C}$ be prestacks on X. Suppose $\mathcal{C}$ is a stack. The prestack $\underline{\mathrm{Hom}}(\mathcal{P},\mathcal{C})$ on X that takes an open set U to the category $\mathrm{Hom}(\mathcal{P}|_U,\mathcal{C}|_U)$ is a stack.*
2. *Let $\mathcal{C}$ be a stack on X. Let c and d be two objects of $\mathcal{C}(X)$. The presheaf of hom sets $U \mapsto \mathrm{Hom}_{\mathcal{C}(U)}(c|_U, d|_U)$ is a sheaf.*
3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two stacks on X. The map $\phi : \mathcal{C} \to \mathcal{D}$ has a left adjoint (resp. has a right adjoint, resp. is an equivalence) if and only if the maps $\phi_x : \mathcal{C}_x \to \mathcal{D}_x$ on stalks all have left adjoints (resp. all have right adjoints, resp. are all equivalences).*
4. *The inclusion 2-functor $\mathrm{St}(X) \to \mathrm{Prest}(X)$ has a right adoint, called* stackification. *Denote the stackification of $\mathcal{P}$ by $\mathcal{P}^\dagger$. The adjunction morphism $\mathcal{P} \to \mathcal{P}^\dagger$ induces an equivalence on stalks.*

A.3. **Operations on stacks.**

Definition A.7. Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. If $\mathcal{C}$ is a prestack on X let $f_*\mathcal{C}$ denote the prestack on Y that associates to an open set U the category $f_*\mathcal{C}(U) := \mathcal{C}\big(f^{-1}(U)\big)$. We call $f_*\mathcal{C}$ the *pushforward* of $\mathcal{C}$.

It is easy to verify that f_* defines a strict 2-functor $\mathrm{Prest}(X) \to \mathrm{Prest}(Y)$, and that if $\mathcal{C}$ is a stack then $f_*\mathcal{C}$ is also. The definition for the pullback of a stack is more complicated – it requires a direct 2-limit over neighborhoods, followed by stackification:

Definition A.8. Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. If $\mathcal{C}$ is a prestack on Y, let $f_p^*\mathcal{C}$ be the prestack on X that associates to an open set

$U \subset X$ the category

$$f_p^*\mathcal{C} = \varinjlim_{V|V \supset f(U)} {}_2\, \mathcal{C}(V)$$

Let $f^*\mathcal{C}$ denote the stackification of the prestack $f_p^*\mathcal{C}$.

Example A.9. If $x \in X$ and $i : \{x\} \hookrightarrow X$ denote the inclusion map, the prestack $i_p^*\mathcal{C}$ on $\{x\}$ coincides with the stalk $\mathcal{C}_x$. Thus, if $f : X \to Y$, we have an equivalence of stalk categories $\left(f_p^*\mathcal{C}\right)_x \cong \mathcal{C}_{f(x)}$.

Example A.10. If $U \subset X$ is open, and $j : U \hookrightarrow X$ denotes the inclusion map, we have $j_p^*\mathcal{C}(V) = \mathcal{C}(V)$ when $V \subset U$ is another open set. If $\mathcal{C}$ is a stack then $j_p^*\mathcal{C}$ is also. We often denote $j^*\mathcal{C}$ by $\mathcal{C}|_U$.

Proposition A.11. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map.*

The 2-functor $f_* : \mathrm{St}(X) \to \mathrm{St}(Y)$ is right 2-adjoint to the 2-functor $f^* : \mathrm{St}(Y) \to \mathrm{St}(X)$.

A.4. **Descent for stacks.** Let X be a topological space, and let $\{U_i\}_{i \in I}$ be a d-cover of X. Suppose we are given the following data:

(0) For each $i \in I$ a stack $\mathcal{C}_i$ on U_i.
(1) For each $i, j \in I$ with $U_j \subset U_i$, an equivalence of stacks $\mathcal{C}_i|_{U_j} \xrightarrow{\sim} \mathcal{C}_j$.
(2) For each $i, j, k \in I$ with $U_k \subset U_j \subset U_i$, an isomorphism between the composite equivalence $\mathcal{C}_i|_{U_j}|_{U_k} \xrightarrow{\sim} \mathcal{C}_j|_{U_k} \xrightarrow{\sim} \mathcal{C}_k$ and $\mathcal{C}_i|_{U_k} \xrightarrow{\sim} \mathcal{C}_k$.
(3) Such that for each $i, j, k, \ell \in I$ with $U_\ell \subset U_k \subset U_j \subset U_i$, the tetrahedron commutes:

$$\begin{tikzcd} \mathcal{C}_i|_{U_j}|_{U_k}|_{U_\ell} \arrow[r] \arrow[d] \arrow[dr] & \mathcal{C}_j|_{U_k}|_{U_\ell} \arrow[d] \\ \mathcal{C}_k|_{U_\ell} \arrow[r] & \mathcal{C}_\ell \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C}_i|_{U_j}|_{U_k}|_{U_\ell} \arrow[r] \arrow[d] & \mathcal{C}_j|_{U_k}|_{U_\ell} \arrow[d] \arrow[dl] \\ \mathcal{C}_k|_{U_\ell} \arrow[r] & \mathcal{C}_\ell \end{tikzcd}$$

We will abuse terminology and refer to such data as a *stack on the d-cover* $\{U_i\}$. Stacks on $\{U_i\}$ form the objects of a 2-category $\mathrm{St}(\{U_i\})$ in the natural way. If $\mathcal{C}$ is a stack on X then $\mathcal{C}_i := \mathcal{C}|_{U_i}$ and the identity 1- and 2-morphisms form a stack on the d-cover $\{U_i\}$, and the assignment $\mathcal{C} \mapsto \{\mathcal{C}_i := \mathcal{C}|_{U_i}\}$ forms a strict 2-functor in a natural way.

Theorem A.12. *The natural restriction 2-functor* $\mathrm{St}(X) \to \mathrm{St}(\{U_i\})$ *is an equivalence of 2-categories.*

Remark A.13. The 2-category $\mathrm{St}(\{U_i\})$ may be interpreted as an inverse limit (or "inverse 3-limit") of the 2-categories $\mathrm{St}(U_i)$. There is a sense then in which the theorem means stacks form a *2-stack*. See [4].

Appendix B. 2-categories

In this appendix we summarize some of the theory of 2-categories, and fix our conventions.

Definition B.1. A *strict 2-category* $\mathbf{C}$ consists of

(1) a collection $\mathrm{Ob}(\mathbf{C})$ of *objects*

(2) for each pair $x, y \in \mathrm{Ob}(\mathbf{C})$ a category $\mathrm{Hom}_{\mathbf{C}}(x,y)$
(3) for each triple $x, y, z \in \mathrm{Ob}(\mathbf{C})$, a *composition functor* $\mathrm{Hom}_{\mathbf{C}}(y,z) \times \mathrm{Hom}_{\mathbf{C}}(x,y) \to \mathrm{Hom}_{\mathbf{C}}(x,z)$.

The composition functors are assumed to satisfy associativity and to have units in the strict sense: for each object $x \in \mathbf{C}$, there is an object 1_x of $\mathrm{Hom}(x,x)$ such that for each y, $\mathrm{Hom}(x,y) \xrightarrow{\circ 1_x} \mathrm{Hom}(x,y)$ and $\mathrm{Hom}(y,x) \xrightarrow{1_x \circ} \mathrm{Hom}(y,x)$ are the identity functors, and such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(z,w) \times \mathrm{Hom}(y,z) \times \mathrm{Hom}(x,y) \arrow[r] \arrow[d] & \mathrm{Hom}(z,w) \times \mathrm{Hom}(x,z) \arrow[d] \\
\mathrm{Hom}(y,z) \times \mathrm{Hom}(x,y) \arrow[r] & \mathrm{Hom}(x,w)
\end{tikzcd}$$

for each x, y, z, w. Objects of $\mathrm{Hom}_{\mathbf{C}}(x,y)$ are called *1-morphisms* of $\mathbf{C}$, and morphisms of $\mathrm{Hom}_{\mathbf{C}}(x,y)$ are called *2-morphisms* of $\mathbf{C}$.

We will also use the following terminology:

Definition B.2. A *(2,1)-category* is a 2-category $\mathbf{C}$ all of whose 2-morphisms are invertible.

Remark B.3. If **cat** denotes the cartesian closed category whose objects are categories, and whose morphisms are functors, then a strict 2-category is a **cat**-enriched category in the sense of [12]. If **gpd** denotes the full subcategory of **cat** whose objects are groupoids, then a $(2,1)$-category is a **gpd**-enriched category.

Example B.4. There is a 2-category **Cat** whose objects are categories, and where the usual categories of functors and natural transformations are the hom categories.

Note that we use a different symbol for the 2-category **Cat** than for the 1-category **cat**. **Cat** is the more natural object.

Remark B.5. There is a more natural notion of *weak* 2-category, where the associativity diagram above is required to commute only up to isomorphism, and these isomorphisms are required to satisfy some equations of their own. Every weak 2-category is equivalent in the appropriate sense to a strict one. Moreover, the 2-categories encountered in this paper are either strict already (such as the 2-category of stacks or of prestacks) or else may be easily made strict by a trick (such as the fundamental 2-groupoid and the exit-path 2-category). We have therefore decided to develop this paper in terms of strict 2-categories.

Definition B.6. Let $\mathbf{C}$ be a 2-category. The *opposite 2-category* $\mathbf{C}^{op}$ is the 2-category with the same objects as $\mathbf{C}$, with $\mathrm{Hom}_{\mathbf{C}^{op}}(x,y) = \mathrm{Hom}_{\mathbf{C}}(y,x)$, and with the evident composition functor.

Remark B.7. One could also define a kind of "opposite 2-category" by reversing only the 2-morphisms, or by reversing both 1- and 2-morphisms. We will not need these variations and so we won't introduce notation for them.

B.1. **Two-dimensional composition in a 2-category.** Let $\mathbf{C}$ be a 2-category. Let x and y be objects in $\mathbf{C}$. If α, β, and γ are three 1-morphisms in $\mathbf{C}$, and $f : \alpha \to \beta$ and $g : \beta \to \gamma$ are 2-morphisms, then we may of course form a third 2-morphism $g \circ f : \alpha \to \beta$ by taking the composition of g and f in the 1-category $\mathrm{Hom}_{\mathbf{C}}(x, y)$.

There is another direction that we may compose 2-morphisms. Let x, y and z be three objects of $\mathbf{C}$, let α and β be two 1-morphisms from x to y and let γ and δ be two 1-morphisms from y to z. Let $f : \alpha \to \beta$ and $h : \gamma \to \delta$ be 2-morphisms. Then we may form a new 2-morphism $h \star f : \gamma \circ \alpha \to \delta \circ \beta$ by applying the functors $\gamma \circ (-) : \mathrm{Hom}_{\mathbf{C}}(x, y) \to \mathrm{Hom}_{\mathbf{C}}(x, z)$ and $(-) \circ \beta : \mathrm{Hom}_{\mathbf{C}}(y, z) \to \mathrm{Hom}_{\mathbf{C}}(x, z)$. That is, let $h \star f$ denote the composite map

$$\gamma \circ \alpha \stackrel{\gamma \circ f}{\to} \gamma \circ \beta \stackrel{h \circ \beta}{\to} \delta \circ \beta$$

Now, let $\mathbf{C}$ be a 2-category, and let x, y, and z be objects of $\mathbf{C}$. Let α, β, and γ be 1-morphisms $x \to y$, and let δ, ϵ, and ζ be 1-morphisms $y \to z$. Let $f : \alpha \to \beta$, $g : \beta \to \gamma$, $h : \delta \to \epsilon$, and $k : \epsilon \to \zeta$ be 2-morphisms. We have the following equation:

$$(k \star g) \circ (h \star f) = (k \circ h) \star (g \circ f)$$

In practice, this equation allows us to ignore the difference between $\circ$ and $\star$ for 2-morphisms. In fact, it follows that given any collection of 2-morphisms that may be composed using $\circ$ and $\star$, any two compositions agree.

B.2. **Adjoints and equivalences within a 2-category.**

Definition B.8. Let $\mathbf{C}$ be a 2-category. Suppose f is a 1-morphism between objects x and y in $\mathbf{C}$. A *right adjoint* to f is a triple (g, η, ϵ), where g is a 1-morphism $y \to x$ called the *adjoint*, $\eta : 1_x \to gf$ and $\epsilon : fg \to 1_y$ are 2-morphisms called the *adjunction morphisms*, and the so-called "triangle identities" hold: the natural maps $\eta g : g \to gfg$ and $g\epsilon : gfg \to g$ compose to 1_g, and the natural maps $f\eta : f \to fgf$ and $\epsilon f : fgf \to f$ compose to 1_f. Dually, (f, ϵ, η) is called a *left adjoint* of g.

We sometimes abuse notation by suppressing the adjunction morphisms ϵ and η.

Proposition B.9. *Let $\mathbf{C}$ be a 2-category, and let f be a 1-morphism in $\mathbf{C}$. If f has a right (resp. left) adjoint (g, α, β), then (g, α, β) is unique up to unique isomorphism commuting with α and β.*

Definition B.10. Let $\mathbf{C}$ be a 2-category, and let $f : x \to y$ be a 1-morphism in $\mathbf{C}$. Then f is called an *equivalence* if it has a right adjoint g, and if the adjunction maps $1 \to fg$ and $gf \to 1$ are both isomorphisms. This is equivalent to requiring f to have a left adjoint g with isomorphisms for adjunction maps.

B.3. **Commutative diagrams in a 2-category.** Recall that, in a 1-category, a commutative diagram is a collection of objects and of morphisms between them such that any two paths of composable arrows in the diagram between objects x and y coincide. In a 2-category, we say that a diagram of objects, morphisms, and 2-morphisms commutes if for each pair of composable paths $f_1, f_2, f_3 \ldots$ and $g_1, g_2, g_3 \ldots$ between objects x and$y$, any two composable sequences of 2-morphisms from $\ldots f_3 \circ f_2 \circ f_1$ to $\ldots g_3 \circ g_2 \circ g_1$ coincide.

Example B.11. Let $\mathbf{C}$ be a 2-category. A *commutative triangle* in $\mathbf{C}$ is a triple x, y, z of objects, a triple $x \to y$, $y \to z$, and $x \to z$ of 1-morphisms, and an isomorphism between the composite $x \to y \to z$ and the map $x \to z$. A *commutative square* is a tuple of objects w, x, y, z, tuple of 1-morphisms $w \to x$, $w \to y$, $x \to z$, $y \to z$, and a 2-isomorphism between the composites $w \to x \to z$ and $w \to y \to z$. For typesetting reasons, we omit the picture of the isomorphism when we draw a commutative triangle or square:

$$\begin{tikzcd} x \arrow[r] \arrow[dr] & y \arrow[d] \\ & z \end{tikzcd} \qquad \begin{tikzcd} w \arrow[r] \arrow[d] & x \arrow[d] \\ y \arrow[r] & z \end{tikzcd}$$

Example B.12. Let $\mathbf{C}$ be a 2-category. A *commutative tetrahedron* in $\mathbf{C}$ is a tuple w, x, y, z of objects, a collection of 1-morphisms $w \to x$, $w \to y$, $w \to z$, $x \to y$, $x \to z$, and a collection of 2-isomorphisms between the composite $w \to x \to y$ and $w \to y$, the composite $w \to x \to z$ and $w \to z$, the composite $w \to y \to z$ and $w \to z$, and the composite $x \to y \to z$ and $x \to z$, such that the two isomorphisms between the composite $w \to x \to z$ and $w \to y \to z$ coincide. We often draw a commutative tetrahedron in the following manner:

$$\begin{tikzcd} w \arrow[r] \arrow[d] \arrow[dr] & x \arrow[d] \\ y \arrow[r] & z \end{tikzcd} = \begin{tikzcd} w \arrow[r] \arrow[d] & x \arrow[d] \arrow[dl] \\ y \arrow[r] & z \end{tikzcd}$$

Remark B.13. We may go on defining commutative n-simplices for $n > 3$. When we refer to a commutative n-simplex in $\mathbf{C}$ we are referring to a diagram in which all the 2-morphisms are isomorphisms. This is most satisfying when $\mathbf{C}$ is a (2,1)-category: in that case the collection of objects, 1-morphisms, commutative triangles, commutative tetrahedra, etc. assemble to a simplicial set called the *nerve* of the (2,1)-category. (Defining the nerve of a general (2,2)-category is more subtle.)

Example B.14. A *prism* with vertices a, b, c, x, y, z is a collection of arrows $a \to b$, $a \to c$, $b \to c$, $a \to x$ $b \to y$, $c \to z$, $x \to y$, $x \to z$, and $y \to z$ and a collection of 2-isomorphisms between $a \to b \to c$ and $a \to c$, between $x \to y \to z$ and $x \to z$, between $a \to b \to y$ and $a \to x \to y$, between $b \to c \to z$ and $b \to y \to z$, and between $a \to c \to z$ and $a \to x \to z$, as in the picture.

$$\begin{tikzcd} a \arrow[rr] \arrow[dr] \arrow[ddd] & & c \arrow[ddd] \\ & b \arrow[ur] \arrow[ddd] & \\ & & \\ x \arrow[rr] \arrow[dr] & & z \\ & y \arrow[ur] & \end{tikzcd}$$

The prism is called commutative if the two isomorphisms between $a \to c \to z$ and $a \to x \to y \to z$ coincide.

B.4. **2-groupoids.**

Definition B.15. A *2-groupoid* is a 2-category in which all the morphisms are equivalences and all the 2-morphisms are isomorphisms.

Remark B.16. A 2-groupoid with one object is called a *2-group*. To each object x in a 2-category we can associate a 2-group $\mathrm{Aut}(x)$ whose unique object is x, whose morphisms are the self-equivalence of x, and whose 2-morphisms are isomorphisms between these self-equivalences.

It is possible to describe 2-groups in terms of more classical algebraic structures. The nerve of a 2-groupoid is a fibrant simplicial set whose homotopy groups (in each connected component) vanish above dimension 2. It is possible to show that connected, pointed spaces X whose homotopy groups vanish above dimension 2 are classified up to homotopy by a group G (the fundamental group of X), a commutative group A (the second homotopy group of X), and an element in group cohomology $H^3(G;A)$ (a Postnikov invariant of X). All this is more naturally encoded in terms of a "crossed module."

B.5. **2-functors between 2-categories.**

Definition B.17. Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. A *2-functor* $F : \mathbf{C} \to \mathbf{D}$ is

(1) An assignment that takes an object $x \in \mathbf{C}$ to an object $Fx \in \mathbf{D}$.
(2) A collection of functors $F : \mathrm{Hom}_{\mathbf{C}}(x, y) \to \mathrm{Hom}_{\mathbf{D}}(Fx, Fy)$.
(3) For each triple x, y, z of objects, a natural isomorphism $\mu_{x,y,z}$ between the two ways of composing the square:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(y,z) \times \mathrm{Hom}_{\mathbf{C}}(x,y) \arrow[r, "{(F,F)}"] \arrow[d] & \mathrm{Hom}_{\mathbf{D}}(Fy,Fz) \times \mathrm{Hom}_{\mathbf{D}}(Fx,Fy) \arrow[d] \\
\mathrm{Hom}_{\mathbf{C}}(x,z) \arrow[r, "F"'] & \mathrm{Hom}_{\mathbf{D}}(Fx,Fz)
\end{tikzcd}$$

We furthermore assume that $F(1_x) = 1_{Fx}$ for each x, and that a certain diagram of μs built from a quadruple w, x, y, z of objects commutes. See [11] for details.

Definition B.18. Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories, and let $F : \mathbf{C} \to \mathbf{D}$ be a 2-functor. F is called *strict* if all the coherence maps $\mu_{x,y,z}$ are identities. The data of a strict 2-functor is equivalent to the data of a **cat**-enriched functor in the sense of [12].

Remark B.19. We will shortly introduce a notion of equivalence for 2-functors (in fact we will introduce a 2-category of 2-functors); note that not all 2-functors are equivalent to strict ones.

Many authors reserve the word "2-functor" for what we have called strict functors, and call 2-functors *pseudofunctors* (e.g. [8], [13]), though usually not in the more recent literature.

Remark B.20. There is a more general notion of 2-functor where the coherence maps μ are not required to be invertible. They are often called "lax 2-functors." There are lax versions of many concepts in 2-category theory, where one replaces an isomorphism in a definition with a map in one direction or another. For our purposes – that is, stacks of categories – the non-lax versions of all these concepts seem to be the correct ones.

Example B.21. Let $\mathbf{C}$ be a 2-category and let $\mathbf{Cat}$ be the 2-category of 1-categories. For each object x of $\mathbf{C}$ there is a strict 2-functor $\mathrm{Hom}(x,-) : \mathbf{C} \to \mathbf{Cat}$ and a strict 2-functor $\mathrm{Hom}(-,x) : \mathbf{C}^{op} \to \mathbf{Cat}$. In fact, there is a strict 2-functor $\mathrm{Hom} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Cat}$.

Proposition B.22. *Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories, and let $F : \mathbf{C} \to \mathbf{D}$ be a 2-functor. If f is a 1-morphism in $\mathbf{C}$, and f has a left (resp. right) adjoint, then Ff has a left (resp. right) adjoint in $\mathbf{D}$. Furthermore if f is an equivalence in $\mathbf{C}$ then Ff is an equivalence in $\mathbf{D}$.*

B.6. **Composite 2-functors.** Let $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ be 2-categories. Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ be 2-functors. There is a *composite 2-functor* $GF = G \circ F$ from $\mathbf{C}$ to $\mathbf{E}$; GF is defined on objects, 1-morphisms, and 2-morphisms of $\mathbf{C}$ in the evident way. To complete the definition it is necessary to describe the coherence data $\mu_{x,y,z}$ for GF – this is straightforward but we refer to [8] for details.

B.7. **The 2-category** $\mathrm{2Funct}(\mathbf{C}, \mathbf{D})$**.** Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. For every pair of 2-functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ we may define a 1-category $\mathrm{2Nat}(F,G)$. Objects of $\mathrm{2Nat}(F,G)$ are called *2-natural transformations*, and morphisms are called *modifications*. Given three 2-functors F, G and H from $\mathbf{C}$ to $\mathbf{D}$, a composition functor $\mathrm{2Nat}(G,H) \times \mathrm{2Nat}(F,G) \to \mathrm{2Nat}(F,H)$ is defined. This composition is strictly associative, and it has strict units, so this data defines a 2-category $\mathrm{2Funct}(\mathbf{C}, \mathbf{D})$. We define 2-natural transformations here; more details may be found in [11]:

Definition B.23. Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. Let F and G be two 2-functors from $\mathbf{C}$ to $\mathbf{D}$. A *2-natural transformation* n from F to G consists of

(1) An assignment that takes objects x of $\mathbf{C}$ to 1-morphisms $n(x) : Fx \to Gx$ in $\mathbf{D}$
(2) An assignment that takes 1-morphisms $f : x \to y$ in $\mathbf{C}$ to isomorphisms $n(f) : Gf \circ n(x) \cong n(x) \circ Ff$.

such that, for each pair of 1-morphisms $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ in $\mathbf{C}$, a certain prism with vertices Fx, Fy, Fz, Gx, Gy, Gz commutes. See [11] for details.

Proposition B.24. *Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. Let $n : F \to G$ be a 1-morphism in* $\mathrm{2Funct}(\mathbf{C}, \mathbf{D})$*.*

(1) *n has a right (resp. left) adjoint $m : F \to G$ if and only if each 1-morphism $n(x) : F(x) \to G(x)$ in $\mathbf{D}$ has a left (resp. right) adoint. (see definition B.8)*
(2) *n is an equivalence if and only if each $n(x) : F(x) \to G(x)$ is an equivalence in $\mathbf{D}$. (see definition B.10)*

B.8. **Adjoint 2-functors between 2-categories.** Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories, and let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be 2-functors. We may form the two 2-functors

$$\mathrm{Hom}_{\mathbf{D}}(F-,-) : \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Cat}$$
$$\mathrm{Hom}_{\mathbf{C}}(-,G-) : \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Cat}$$

Definition B.25. Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. A *2-adjunction* from $\mathbf{C}$ to $\mathbf{D}$ is a pair of 2-functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ together with a 2-natural equivalence – i.e. an equivalence in the 2-category $\mathrm{2Funct}(\mathbf{C}^{op} \times \mathbf{D}, \mathbf{Cat})$ – between the two 2-functors $\mathrm{Hom}(F-,-)$ and $\mathrm{Hom}(-,G-)$.

We say that F is *left 2-adjoint* to G and that G is *right 2-adjoint* to F.

Remark B.26. This is another definition with lax generalizations.

We can apply the adjunction to the identity 2-functors $Fc \to Fc$ and $Gd \to Gd$ to obtain 2-natural transformations $1_{\mathbf{C}} \to GF$ and $FG \to 1_{\mathbf{D}}$.

Definition B.27. Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. An *equivalence* from $\mathbf{C}$ to $\mathbf{D}$ is a pair of adjoint 2-functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ such that the 2-natural transformations $1_{\mathbf{C}} \to GF$ and $FG \to 1_{\mathbf{D}}$ are equivalences.

A 2-functor $F : \mathbf{C} \to \mathbf{D}$ is called *essentially fully faithful* if the functor $\mathrm{Hom}_{\mathbf{C}}(x, y) \to \mathrm{Hom}_{\mathbf{D}}(Fx, Fy)$ is an equivalence of categories for every pair of objects $x, y \in \mathbf{C}$. F is called *essentially surjective* if for every object $d \in \mathbf{D}$ there is an object $c \in \mathbf{D}$ such that Fc and d are equivalent in $\mathbf{D}$.

Proposition B.28. *Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. Let $F : \mathbf{C} \to \mathbf{D}$ be a 2-functor. The following are equivalent.*

1. *F is essentially fully faithful and essentially surjective.*
2. *F is part of an equivalence $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$.*

B.9. **Direct and inverse 2-limits of 1-categories.** Let $\mathbf{I}$ be a 2-category, and let $F : \mathbf{I} \to \mathbf{Cat}$ be a 2-functor. It is possible to define categories $2\varprojlim_{\mathbf{I}} F$ and $2\varinjlim_{\mathbf{I}} F$ with the appropriate 2-universal property. We will give these definitions here in the form that we need them; in particular we only define $2\varinjlim_{\mathbf{I}} F$ in case $\mathbf{I}$ is a filtered poset.

Definition B.29. Let $\mathbf{I}$ be a 2-category and let $F : \mathbf{I} \to \mathbf{Cat}$ be a 2-functor. Let $2\varprojlim_I F$ denote the category whose objects consist of the following data:

1. An assignment that takes an object $i \in \mathbf{I}$ to an object $x_i \in F(i)$.
2. An assignment that takes a morphism $f : i \to j$ to an isomorphism $F(f)(x_i) \cong x_j$.

We require that for each commutative triangle $\alpha : gf \cong h$ with vertices i, j, k in $\mathbf{I}$, the diagram

$$\begin{tikzcd} F(g)F(f)x_i \arrow[r] \arrow[d, "\alpha"] & F(g)x_j \arrow[d] \\ F(h) \arrow[r] & x_k \end{tikzcd}$$

commutes. The morphisms of this category are collections of maps $x_i \to y_i$ commuting with the maps $F(f)(x_i) \to x_j$.

Definition B.30. Let I be a filtered poset, and let $F : I \to \mathbf{Cat}$ be a 2-functor. Let $2\varinjlim_I F$ denote the category whose objects are $\coprod_{i \in I} F(i)$, and whose morphisms $\mathrm{Hom}(x \in F(i), y \in F(j))$ are elements of the limit

$$\varinjlim_{k \geq i \text{ and } j} \mathrm{Hom}_{F(k)}(x_i|_k, x_j|_k)$$

Here if $\ell \leq k$ and $x \in F(\ell)$, the notation $x|_k$ denotes the image of x under the functor $F(\ell) \to F(k)$ induced by the unique morphism $\ell \to k$.

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